# Max-size popular matchings and extensions 

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#### Abstract

We consider the max-size popular matching problem in a roommates instance $G=(V, E)$ with strict preference lists. A matching $M$ is popular if there is no matching $M^{\prime}$ in $G$ such that the vertices that prefer $M^{\prime}$ to $M$ outnumber those that prefer $M$ to $M^{\prime}$. We show it is NP-hard to compute a max-size popular matching in $G$. This is in contrast to the tractability of this problem in bipartite graphs where a max-size popular matching can be computed in linear time. We define a subclass of max-size popular matchings called strongly dominant matchings and show a linear time algorithm to solve the strongly dominant matching problem in a roommates instance. We consider a generalization of the max-size popular matching problem in bipartite graphs: this is the max-weight popular matching problem where there is also a weight function $w: E \rightarrow \mathbb{R}$ and we seek a popular matching of largest weight. We show this is an NP-hard problem and this is so even when $w(e) \in\{1,2\}$ for every $e \in E$. We also show an algorithm with running time $O^{*}\left(2^{n / 4}\right)$ to find a max-weight popular matching matching in $G=(A \cup B, E)$ on $n$ vertices.


## 1 Introduction

Consider a matching problem in $G=(A \cup B, E)$ where each vertex has a strict ranking of its neighbors. The goal is to find an optimal way of pairing up vertices: stability is the usual notion of optimality in such a setting. A matching $M$ is stable if $M$ admits no blocking edge, i.e., an edge ( $a, b$ ) such that $a$ and $b$ prefer each other to their respective assignments in $M$. Stable matchings always exist in $G$ and can be computed in linear time [11].

In applications such as matching students to advisers or applicants to training posts, we would like to replace the notion of "no blocking edges" with a more relaxed notion of "global stability" for the sake of obtaining a matching that is better in a social sense, for instance, a matching of larger size. For this, we need to formalize the notion of a globally stable matching; roughly speaking, a globally stable matching should be one such that there is no matching where more people are happier.

This is precisely the notion of popularity introduced by Gärdenfors [13 in 1975. We say a vertex $u \in A \cup B$ prefers matching $M$ to matching $M^{\prime}$ if either (i) $u$ is matched in $M$ and unmatched in $M^{\prime}$ or (ii) $u$ is matched in both $M, M^{\prime}$ and $u$ prefers $M(u)$ to $M^{\prime}(u)$. For any two matchings $M$ and $M^{\prime}$, let $\phi\left(M, M^{\prime}\right)$ be the number of vertices that prefer $M$ to $M^{\prime}$.

Definition 1. A matching $M$ is popular if $\phi\left(M, M^{\prime}\right) \geq \phi\left(M^{\prime}, M\right)$ for every matching $M^{\prime}$ in $G$, i.e., $\Delta\left(M, M^{\prime}\right) \geq 0$ where $\Delta\left(M, M^{\prime}\right)=\phi\left(M, M^{\prime}\right)-\phi\left(M^{\prime}, M\right)$.

In an election between $M$ and $M^{\prime}$ where vertices cast votes, $\phi\left(M, M^{\prime}\right)$ is the number of votes for $M$ versus $M^{\prime}$ and $\phi\left(M^{\prime}, M\right)$ is the number of votes for $M^{\prime}$ versus $M$. A popular matching never loses an election to another matching: thus it is a weak Condorcet winner [4] in the corresponding voting instance. Although (weak) Condorcet winners need not exist in a general voting instance, popular matchings always exist in a bipartite graph with strict preference lists, since every stable matching is popular [13].

All stable matchings match the same subset of vertices 12 and the size of a stable matching can be as low as $\left|M_{\max }\right| / 2$, where $M_{\max }$ is a max-size matching in $G$. One of the main motivations to relax stability to popularity is to obtain larger matchings and it is known that a max-size popular matching has size at least $2\left|M_{\max }\right| / 3$. Polynomial time algorithms to compute a max-size popular matching in $G=(A \cup B, E)$ are known [1622].

A roommates instance is a graph $G=(V, E)$, that is not necessarily bipartite, with strict preference lists. Stable matchings need not always exist in $G$ and there are several polynomial time algorithms 192627 to determine if a stable matching exists or not. The definition of popularity (Definition 1) carries over to roommates instances - though popular matchings need not exist in $G$, popular mixed matchings, i.e., probability distribution over matchings, always exist in $G$ and can be efficiently computed [21. Observe that stable mixed matchings need not always exist in a roommates instance (see the Appendix).

We currently do not know the complexity of the popular matching problem in a roommates instance, i.e., does $G$ admit a popular matching? The complexity of finding a max-size popular matching in $G$ was also an open problem so far and we show its hardness here.

Theorem 1. The max-size popular matching problem in a roommates instance $G=(V, E)$ with strict preference lists is NP-hard.

We show the above problem is NP-hard even in instances that admit stable matchings. Note that a stable matching is a min-size popular matching [16]. All the polynomial time algorithms that compute popular matchings in bipartite graphs [11|6|22] compute either stable matchings or dominant matchings. A popular matching $M$ is dominant if $M$ is more popular than every larger matching [7], thus $M$ is a max-size popular matching.

Though the name "dominant" was given to this class of matchings in [7], dominant matchings in bipartite graphs first appeared in [16, which gave the first polynomial time max-size popular matching algorithm in bipartite graphs. More precisely, a matching satisfying Definition 2 was constructed in the given bipartite graph. For any matching $M$, call an edge ( $u, v$ ) negative to $M$ if both $u$ and $v$ prefer their assignments in $M$ to each other.

Definition 2. A matching $M$ is strongly dominant in $G=(V, E)$ if there is a partition $(L, R)$ of the vertex set $V$ such that (i) $M \subseteq L \times R$, (ii) $M$ matches all vertices in $R$, (iii) every blocking edge to $M$ is in $R \times R$, and (iv) every edge in $L \times L$ is negative to $M$.

Consider the complete graph on 4 vertices $a, b, c, d$ where $a$ 's preference list is $b \succ c \succ d$ (i.e., top choice $b$, followed by $c$ and then $d$ ), $b$ 's preference list is $c \succ a \succ d$, $c$ 's preference list is $a \succ b \succ d$, and $d$ 's preference list is $a \succ b \succ c$. This instance has no stable matching. $M_{1}=\{(a, d),(b, c)\}$ and $M_{2}=\{(a, c),(b, d)\}$ are two strongly dominant matchings here: the corresponding partitions are $L_{1}=\{b, d\}, R_{1}=\{a, c\}$ and $L_{2}=\{c, d\}, R_{2}=\{a, b\}$.

A strongly dominant matching $M$ with vertex partition $(L, R)$ is an $R$-perfect stable matching in the bipartite graph with $L$ on the left, $R$ on the right and edge set $E \cap(L \times R)$. It is also important to note that any blocking edge to $M$ in $G$ is in $R \times R$ and all edges in $L \times L$ are negative to $M$.

It was shown in [7] that a popular matching $M$ is dominant if and only if there is no augmenting path with respect to $M$ in the subgraph obtained by deleting all negative edges with respect to M. In bipartite graphs, dominant matchings and strongly dominant matchings are equivalent [7]. Moreover, such a matching always exists in a bipartite graph and can be computed in linear time 22.

It can also be shown [22] that every strongly dominant matching in $G=(V, E)$ is dominant. However in non-bipartite graphs, not every dominant matching is strongly dominant. The complexity of the dominant matching problem in a roommates instance is currently not known. Here we efficiently solve the strongly dominant matching problem.

Theorem 2. There is a linear time algorithm to determine if an instance $G=(V, E)$ with strict preference lists admits a strongly dominant matching or not and if so, return one.

### 1.1 Bipartite instances

A natural generalization of the max-size popular matching problem in a bipartite instance $G=$ $(A \cup B, E)$ is the max-weight popular matching problem, where there is a weight function $w: E \rightarrow \mathbb{R}$ and we seek a popular matching of largest weight. Several natural popular matching problems can be formulated with the help of edge weights: these include computing a popular matching with as
many of our "favorite edges" as possible or an egalitarian popular matching (one that minimizes the sum of ranks of partners of all vertices). Thus a max-weight popular matching problem is a generic problem that captures several optimization problems in popular matchings.

The max-weight stable matching problem is well-studied and there are several polynomial time algorithms 8 $8|10| 20|25| 27 \mid 28$ to compute such a matching or its variants in a bipartite graph with strict preference lists. We show the following result here.

Theorem 3. The max-weight popular matching problem in $G=(A \cup B, E)$ with strict preference lists and a weight function $w: E \rightarrow\{1,2\}$ is NP-hard.

A 2-approximate max-weight popular matching in $G=(A \cup B, E)$ with strict preference lists and non-negative edge weights can be computed in polynomial time. We also show the following fast exponential time algorithm, where $n=|A \cup B|$.

Theorem 4. A max-weight popular matching in $G=(A \cup B, E)$ with strict preference lists and a weight function $w: E \rightarrow \mathbb{R}$ can be computed in $O^{*}\left(c^{n}\right)$ time, where $c=2^{1 / 4} \approx 1.19$.

### 1.2 Background and Related results

Algorithmic questions for popular matchings were first studied in the domain of one-sided preference lists [1] in a bipartite instance $G=(A \cup B, E)$ where it is only vertices in $A$ that have preferences over their neighbors. Popular matchings need not always exist here, however popular mixed matchings always exist [21]. This proof extends to the domain of two-sided preference lists (with ties) and to non-bipartite graphs.

Popular matchings always exist in $G=(A \cup B, E)$ with two-sided strict preference lists. An $O\left(m n_{0}\right)$ algorithm to compute a max-size popular matching here was shown in [16, where $m=|E|$ and $n_{0}=\min (|A|,|B|)$. A linear time algorithm for the max-size popular matching problem in such an instance $G$ was shown in [22].

A linear time algorithm was shown in [7] to determine if there is a popular matching in $G=$ $(A \cup B, E)$ that contains a given edge $e$. It was also shown in 7 that dominant matchings in $G=(A \cup B, E)$ are equivalent to stable matchings in a larger bipartite graph. This equivalence implies a polynomial time algorithm to solve the max-weight popular matching problem in a complete bipartite graph.

A description of the popular half-integral matching polytope of $G=(A \cup B, E)$ with strict preference lists was given in [23]. It was shown in [18] that the popular fractional matching polytope (from [21]) for such an instance $G=(A \cup B, E)$ is half-integral. The half-integrality of the popular fractional matching polytope also holds for roommates instances [18].

When preference lists admit ties, the problem of determining if a bipartite instance admits a popular matching or not is NP-hard [26]. It is NP-hard to compute a least unpopularity factor matching in a roommates instance [17]. It was shown in [18] that it is NP-hard to compute a maxweight popular matching problem in a roommates instance with strict preference lists and it is UGC-hard to compute a $\Theta(1)$-approximation.

The complexity of finding a max-weight popular matching in a bipartite instance with strict preference lists was left as an open problem in 18 . This problem along with the complexity of finding a max-size popular matching in a roommates instance are two of the three open problems in popular matchings listed in [5] and we answer these two questions here.

### 1.3 Techniques

Our results are based on LP-duality. Every popular matching $M$ in an instance $G=(V, E)$ is a max-cost perfect matching in the graph $G$ with self-loops added and with edge costs given by a function $\operatorname{cost}_{M}$ (these costs depend on the matching $M$ ). Any optimal solution to the dual LP will be called a witness to M's popularity.
Our hardness results. Witnesses for popular matchings in bipartite graphs first appeared in [21] and they were used in [23]18]3]. Roughly speaking, these algorithms dealt with matchings that had an
element in $\{ \pm 1\}^{n}$ as a witness. Note that a stable matching has $0^{n}$ as a witness. For general popular matchings, there is no such "parity agreement" among the coordinates of any witness and we use this to show that the max-weight popular matching problem in bipartite graphs is NP-hard.

All max-size popular matchings in a bipartite instance match the same subset of vertices [15], however the rural hospitals theorem does not necessarily hold for max-size popular matchings in roommates instances. Such an instance forms the main gadget in the proof of NP-hardness for max-size popular matchings in a roommates instance.

Our algorithms. We generalize the max-size popular matching algorithm for bipartite graphs [22] to solve the strongly dominant matching problem in all graphs. We show a surprisingly simple reduction from the strongly dominant matching problem in $G=(V, E)$ to the stable matching problem in a new roommates instance $G^{\prime}=\left(V, E^{\prime}\right)$. Thus Irving's stable matching algorithm [19] in $G^{\prime}$ solves our problem in linear time.

Our reduction is similar to an analogous correspondence in the bipartite case from [7]. However the graph $G^{\prime}$ in [7] on $3|A|+|B|$ vertices is asymmetric with respect to vertices in $A$ and $B$ of $G=(A \cup B, E)$. Now our new graph $G^{\prime}$ may be regarded as the bidirected version of $G$, i.e., each edge $(u, v)$ in $G$ is replaced by two edges $\left(u^{+}, v^{-}\right)$and $\left(u^{-}, v^{+}\right)$in $G^{\prime}$.

Our fast exponential time algorithm for max-weight popular matchings in $G=(A \cup B, E)$ formulates the convex hull $\mathcal{P}(\boldsymbol{r})$ of all popular matchings with at least 1 witness whose parities agree with a given vector $\boldsymbol{r} \in\{0,1\}^{k}$. Here $k$ is the number of components of size 4 or more in a subgraph $F_{G}$, whose edge set is the union of all popular matchings in $G$.

Our formulation of $\mathcal{P}(\boldsymbol{r})$ is based on the popular fractional matching polytope $\mathcal{P}_{G}$ [21]. We use $\boldsymbol{r}$ to tighten several of the constraints in the formulation of $\mathcal{P}_{G}$ and introduce new variables sandwiched between 0 and 1 to denote fractional parities and formulate a polytope. We use methods from [18|27] along with some new ideas to show that our polytope is integral, more precisely, it is $\mathcal{P}(\boldsymbol{r})$. This leads to the $O^{*}\left(2^{k}\right)$ (where $k \leq n / 4$ ) algorithm.
Organization of the paper. Witness vectors for popular matchings in bipartite and roommates instances are defined in Section 2, Our algorithm for the strongly dominant matching problem in a roommates instance $G=(V, E)$ is given in Section 3. Section 4 shows that finding a max-size popular matching in $G$ is NP-hard. Section 5 shows the NP-hardness of the max-weight popular matching problem in a bipartite instance $G=(A \cup B, E)$. Our fast exponential time algorithm for this problem is given in Section 6.

## 2 Witness of a popular matching

Let $M$ be any matching in our input instance $G=(V, E)$. In order to determine if $M$ is popular or not, we need to check if $\Delta(N, M) \leq 0$ for all matchings $N$ in $G$ (see Definition 1). Computing $\max _{N} \Delta(N, M)$ reduces to computing a max-cost perfect matching in a graph $\tilde{G}$ with edge costs that are defined below.

The graph $\tilde{G}$ is the graph $G$ augmented with self-loops - we assume that every vertex is at the bottom of its own preference list. Adding self-loops allows us to view any matching $M$ in $G$ as a perfect matching $\tilde{M}$ in $\tilde{G}$ by including self-loops for all vertices left unmatched in $M$. We now define a function $\operatorname{cost}_{M}$ on the edge set of $\tilde{G}$. For any edge $(u, v) \in E$, define:

$$
\operatorname{cost}_{M}(u, v)= \begin{cases}2 & \text { if }(u, v) \text { is a blocking edge to } \tilde{M} \\ -2 & \text { if }(u, v) \text { is a negative edge to } \tilde{M} \\ 0 & \text { otherwise }\end{cases}
$$

Recall that an edge $(u, v)$ is negative to $\tilde{M}$ if both $u$ and $v$ prefer their partners in $\tilde{M}$ to each other. Thus $\operatorname{cost}_{M}(u, v)$ is the sum of votes of $u$ and $v$ for each other over $\tilde{M}(u)$ and $\tilde{M}(v)$, respectively, where for any vertex $u$ and neighbors $v, v^{\prime}$ of $u$ : $u$ 's vote for $v$ versus $v^{\prime}$ is 1 if $u$ prefers $v$ to $v^{\prime}$, it is -1 if $u$ prefers $v^{\prime}$ to $v$, else it is 0 (i.e. $v=v^{\prime}$ ). Observe that $\operatorname{cost}_{M}(u, v)=0$ for any edge $(u, v) \in M$.

We now define $\operatorname{cost}_{M}$ for self-loops as well. For any $u \in V, \operatorname{cost}_{M}(u, u)=0$ if $u$ is unmatched in $M$, else $\operatorname{cost}_{M}(u, u)=-1$. Thus $\operatorname{cost}_{M}(u, u)$ is $u$ 's vote for itself versus $\tilde{M}(u)$.

Claim 1 For any matching $N$ in $G, \Delta(N, M)=\operatorname{cost}_{M}(\tilde{N})$.
Proof. We will use the function vote $(\cdot, \cdot)$ here. For any vertex $u$ and neighbors $v, v^{\prime}$ of $u$ in $\tilde{G}$, recall that $\operatorname{vote}_{u}\left(v, v^{\prime}\right)$ is 1 if $u$ prefers $v$ to $v^{\prime}$, it is -1 if $u$ prefers $v^{\prime}$ to $v$, it is 0 otherwise (i.e., $v=v^{\prime}$ ).

Let $\tilde{M}(u)$ be $u$ 's partner in $\tilde{M}$. Observe that $\operatorname{cost}_{M}(a, b)=\operatorname{vote}_{a}(b, \tilde{M}(a))+\operatorname{vote}_{b}(a, \tilde{M}(b))$. Also, $\operatorname{cost}_{M}(u, u)=\operatorname{vote}_{u}(u, \tilde{M}(u))$. We have the following equality from the definitions of $\Delta(\cdot, \cdot)$ and $\operatorname{vote}(\cdot, \cdot)$.

$$
\begin{aligned}
\Delta(N, M) & =\sum_{u \in A \cup B} \operatorname{vote}_{u}(\tilde{N}(u), \tilde{M}(u)) \\
& =\sum_{(a, b) \in N}\left(\operatorname{vote}_{a}(b, \tilde{M}(a))+\operatorname{vote}_{b}(a, \tilde{M}(b))\right)+\sum_{(u, u) \in \tilde{N}} \operatorname{vote}_{u}(u, \tilde{M}(u)) .
\end{aligned}
$$

This sum is exactly $\sum_{(a, b) \in N} \operatorname{cost}_{M}(a, b)+\sum_{(u, u) \in \tilde{N}} \operatorname{cost}_{M}(u, u)$, which is $\operatorname{cost}_{M}(\tilde{N})$.
Thus $M$ is popular if and only if every perfect matching in $\tilde{G}$ has cost at most 0 .

### 2.1 Bipartite instances

Let $G=(A \cup B, E)$ be a bipartite instance with strict preference lists. Consider the max-cost perfect matching LP in $\tilde{G}$ : this is LP1 given below. The set $\tilde{E}$ is the edge set of $\tilde{G}$ and $\tilde{E}(u)$ is the set of edges incident to $u$ in $\tilde{G}$. The linear program LP2 is the dual of LP1.

$$
\begin{array}{ccrc}
\max \sum_{e \in \tilde{E}} \operatorname{cost}_{M}(e) \cdot x_{e} & (\mathrm{LP} 1) & \min \sum_{u \in A \cup B} \alpha_{u} & \text { (LP2) } \\
\text { s.t. } & \sum_{e \in \tilde{E}(u)} x_{e}=1 & \forall u \in A \cup B & \text { s.t. } \\
x_{e} \geq 0 & \forall e \in \tilde{E} . & \alpha_{a}+\alpha_{b} \geq \operatorname{cost}_{M}(a, b) & \forall(a, b) \in E \\
& \alpha_{u} \geq \operatorname{cost}_{M}(u, u) & \forall u \in A \cup B .
\end{array}
$$

$M$ is popular if and only if the optimal value of LP1 is at most 0 (by Claim 1); in fact, the optimal value is exactly 0 since $\tilde{M}$ is a perfect matching in $\tilde{G}$ and $\operatorname{cost}_{M}(\tilde{M})=0$. Thus $M$ is popular if and only if the optimal value of LP2 is 0 (by LP-duality).

Definition 3. For any popular matching M, an optimal solution $\boldsymbol{\alpha}$ to LP2 above is called a witness of $M$.

A popular matching $M$ may have several witnesses. For any witness $\boldsymbol{\alpha}, \sum_{u \in A \cup B} \alpha_{u}=0$ since $\boldsymbol{\alpha}$ is an optimal solution to LP2 and the optimal value of LP2 is 0 . Let $n=|A \cup B|$.

Lemma 1 ([23]). Every popular matching $M$ in $G=(A \cup B, E)$ has a witness in $\{0, \pm 1\}^{n}$.

### 2.2 Roommates instances

Here our input is a graph $G=(V, E)$ with strict preference lists. As done earlier, we will formulate the max-cost perfect matching problem in $\tilde{G}$ with cost function $\operatorname{cost}_{M}$ as our primal LP.

The dual LP (LP3 given below) will be useful to us. Here $\Omega$ is the collection of all odd subsets $S$ of $V$ of size at least 3 and $E[S]$ is the set of all $(u, v) \in E$ such that $u, v \in S$. A matching $M$ is popular in $G=(V, E)$ if and only if the optimal value of LP3 is 0 .

$$
\begin{equation*}
\operatorname{minimize} \sum_{u \in V} \alpha_{u}+\sum_{S \in \Omega}\lfloor|S| / 2\rfloor \cdot z_{S} \tag{LP3}
\end{equation*}
$$

subject to

$$
\begin{aligned}
& \alpha_{u}+\alpha_{v}+\sum_{\substack{S \in \Omega \\
u, v \in S}} z_{S} \geq \operatorname{cost}_{M}(u, v) \forall(u, v) \in E \\
& z_{S} \geq 0 \forall S \in \Omega \quad \text { and } \quad \alpha_{u} \geq \operatorname{cost}_{M}(u, u) \forall u \in V
\end{aligned}
$$

Definition 4. For any popular matching $M$, an optimal solution ( $\boldsymbol{\alpha}, \boldsymbol{z}$ ) to LP3 is called a witness of $M$.

For any witness $(\boldsymbol{\alpha}, \boldsymbol{z})$, we have $\sum_{u \in V} \alpha_{u}+\sum_{S \in \Omega}\lfloor|S| / 2\rfloor \cdot z_{S}=0$. Note that any stable matching in $G$ has $(\mathbf{0}, \mathbf{0})$ as witness.

Theorem 5 gives a characterization of strongly dominant matchings in terms of a special witness $(\boldsymbol{\alpha}, \boldsymbol{z})$. We will use this characterization of strongly dominant matchings in Section 3 .

Theorem 5. A matching $M$ is strongly dominant in $G$ if and only if there exists a feasible solution $(\boldsymbol{\alpha}, \boldsymbol{z})$ to LP3 such that $\alpha_{u}= \pm 1$ for all vertices $u$ matched in $M, \alpha_{u}=0$ for all $u$ unmatched in $M$, $z_{S}=0$ for all $S \in \Omega$, and $\sum_{u \in V} \alpha_{u}=0$.

Proof. Let $M$ be a strongly dominant matching in $G=(V, E)$. So $V$ can be partitioned into $L \cup R$ such that properties (i)-(iv) in Definition 2 are satisfied. Set $z_{S}=0$ for all $S \in \Omega$. We will now construct $\boldsymbol{\alpha}$ as follows. For $u \in V$ :

- if $u \in R$ then set $\alpha_{u}=1$
- else if $u$ is matched in $M$ then set $\alpha_{u}=-1$ else set $\alpha_{u}=0$.

Since $M$ matches all vertices in $R$, all vertices unmatched in $M$ are in $L$. Thus $\alpha_{u}=0$ for all $u$ unmatched in $M$ and $\alpha_{u}= \pm 1$ for all $u$ matched in $M$. For any edge $(u, v) \in M$, since $M \subseteq L \times R$, $\left(\alpha_{u}, \alpha_{v}\right) \in\{(1,-1),(-1,1)\}$ and so $\alpha_{u}+\alpha_{v}=0$. Thus $\sum_{u \in V} \alpha_{u}=0$.

We will now show that $(\boldsymbol{\alpha}, \mathbf{0})$ satisfies the constraints of LP3. We have $\alpha_{u} \geq \operatorname{cost}_{M}(u, u)$. This is because $\alpha_{u}=0=\operatorname{cost}_{M}(u, u)$ for $u$ left unmatched in $M$ and $\alpha_{u} \geq-1=\operatorname{cost}_{M}(u, u)$ for $u$ matched in $M$. We will now show that all edge covering constraints are obeyed.

- Since $\operatorname{cost}_{M}(e) \leq 2$ for any edge $e$ and $\alpha_{u}=1$ for all $u \in R$, all edges in $R \times R$ are covered.
- We also know that any edge in $L \times L$ is a negative edge to $M$, i.e., $\operatorname{cost}_{M}(u, v)=-2$ for any edge $(u, v) \in L \times L$. Since $\alpha_{u} \geq-1$ for any $u \in L$, edges in $L \times L$ are covered.
- We also know that all blocking edges to $M$ are in $R \times R$ and so $\operatorname{cost}_{M}(u, v) \leq 0$ for all $(u, v) \in$ $L \times R$. Since $\alpha_{u} \geq-1$ and $\alpha_{v}=1$, all edges in $L \times R$ are covered.

We will now show the converse. Let $M$ be a matching with a witness $(\boldsymbol{\alpha}, \boldsymbol{0})$ as given in the statement of the theorem. To begin with, $M$ is popular since the objective function of LP3 evaluates to 0 . We will now show that $M$ is strongly dominant.

For that, we will obtain a partition $(L, R)$ of $V$ as follows: $R=\left\{u: \alpha_{u}=1\right\}$ and $L=\{u$ : $\alpha_{u}$ is either 0 or -1$\}$. Since $\tilde{M}$ is an optimal solution of the max-cost perfect matching problem in $\tilde{G}$, complementary slackness conditions imply that if $(u, v) \in M$ then $\alpha_{u}+\alpha_{v}=\operatorname{cost}_{M}(u, v)=0$. Since $u, v$ are matched, $\alpha_{u}, \alpha_{v} \in\{ \pm 1\}$; so one of $u, v$ is in $L$ and the other is in $R$. Thus $M \subseteq L \times R$.

We have $\operatorname{cost}_{M}(u, v) \leq \alpha_{u}+\alpha_{v}$ for every $(u, v) \in E$. There cannot be any edge between 2 vertices left unmatched in $M$ as that would contradict $M$ 's popularity. So $\operatorname{cost}_{M}(u, v) \leq-1$ for all $(u, v) \in E \cap(L \times L)$. Since $\operatorname{cost}_{M}(u, v) \in\{0, \pm 2\}, \operatorname{cost}_{M}(u, v)=-2$ for all edges $(u, v)$ in $L \times L$. In other words, every edge in $L \times L$ is negative to $M$.

Moreover, any blocking edge can be present only in $R \times R$ since $\operatorname{cost}_{M}(u, v) \leq 1$ for all edges $(u, v) \in L \times R$. Finally, since $\alpha_{u}=\operatorname{cost}_{M}(u, u)=0$ for all $u$ unmatched in $M$ (by complementary slackness conditions on LP3 and every vertex $u \in R$ satisfies $\alpha_{u}=1$, it means that all vertices in $R$ are matched in $M$.

## 3 Strongly dominant matchings

In this section we show an algorithm to determine if a roommates instance $G=(V, E)$ admits a strongly dominant matching or not. We will build a new roommates instance $G^{\prime}=\left(V, E^{\prime}\right)$ and show that any stable matching in $G^{\prime}$ can be projected to a strongly dominant matching in $G$ and conversely, any strongly dominant matching in $G=(V, E)$ can be mapped to a stable matching in $G^{\prime}$.

The vertex set of $G^{\prime}$ is the same as that of $G$. Though there is only one copy of each vertex $u$ in $G^{\prime}$, for every $(u, v) \in E$, there will be 2 parallel edges in $G^{\prime}$ between $u$ and $v$; we will call one of these edges $\left(u^{+}, v^{-}\right)$and the other $\left(u^{-}, v^{+}\right)$. Thus $E^{\prime}=\left\{\left(u^{+}, v^{-}\right),\left(u^{-}, v^{+}\right):(u, v) \in E\right\}$. A vertex $v$ appears in 2 forms, as $v^{+}$and $v^{-}$, to each of its neighbors.

We will now define preference lists in $G^{\prime}$. For any $u \in V$, if $u$ 's preference list in $G$ is $v_{1} \succ v_{2} \succ$ $\cdots \succ v_{k}$ then $u$ 's preference list in $G^{\prime}$ is $v_{1}^{-} \succ v_{2}^{-} \succ \cdots \succ v_{k}^{-} \succ v_{1}^{+} \succ v_{2}^{+} \succ \cdots \succ v_{k}^{+}$. Thus $u$ prefers any neighbor in "- form" to any neighbor in "+ form".

As an example, consider the roommates instance on 4 vertices $a, b, c, d$ described in Section 1 , where $d$ was the least preferred vertex of $a, b, c$. Preference lists in the instance $G^{\prime}$ are as follows:

$$
\begin{array}{ll}
a: b^{-} \succ c^{-} \succ d^{-} \succ b^{+} \succ c^{+} \succ d^{+} & b: c^{-} \succ a^{-} \succ d^{-} \succ c^{+} \succ a^{+} \succ d^{+} \\
c: a^{-} \succ b^{-} \succ d^{-} \succ a^{+} \succ b^{+} \succ d^{+} & d: a^{-} \succ b^{-} \succ c^{-} \succ a^{+} \succ b^{+} \succ c^{+}
\end{array}
$$

- A matching $M^{\prime}$ in $G^{\prime}$ is a subset of $E^{\prime}$ such that for each $u \in V, M^{\prime}$ contains at most one edge incident to $u$, i.e., at most one edge in $\left\{\left(u^{+}, v^{-}\right),\left(u^{-}, v^{+}\right): v \in \operatorname{Nbr}(u)\right\}$ is in $M^{\prime}$, where $\operatorname{Nbr}(u)$ is the set of $u$ 's neighbors in $G$.
- For any matching $M^{\prime}$ in $G^{\prime}$, define the projection $M$ of $M^{\prime}$ as follows:

$$
M=\left\{(u, v):\left(u^{+}, v^{-}\right) \text {or }\left(u^{-}, v^{+}\right) \text {is in } M^{\prime}\right\} .
$$

It is easy to see that $M$ is a matching in $G$.
Definition 5. A matching $M^{\prime}$ is stable in $G^{\prime}$ if for every edge $\left(u^{+}, v^{-}\right) \in E^{\prime} \backslash M^{\prime}$ : either (i) $u$ is matched in $M^{\prime}$ to a neighbor ranked better than $v^{-}$or (ii) $v$ is matched in $M^{\prime}$ to a neighbor ranked better than $u^{+}$.

We now present our algorithm to find a strongly dominant matching in $G=(V, E)$.

1. Build the corresponding roommates instance $G^{\prime}=\left(V, E^{\prime}\right)$.
2. Run Irving's stable matching algorithm in $G^{\prime}$.
3. If a stable matching $M^{\prime}$ is found in $G^{\prime}$ then return the projection $M$ of $M^{\prime}$.

Else return " $G$ has no strongly dominant matching".
In Irving's algorithm in $G^{\prime}$, it is possible that a vertex $u$ proposes to some neighbor $v_{t}$ twice: first to $v_{t}^{-}$and later to $v_{t}^{+}$. During $u$ 's first proposal, $v_{t}$ receives a proposal from $u^{+}$and during $u$ 's second proposal, $v_{t}$ receives a proposal from $u^{-}$. We describe Irving's stable matching algorithm [19] along with an example in the Appendix.

We will now prove the correctness of the above algorithm. We will first show that if our algorithm returns a matching $M$, then $M$ is a strongly dominant matching in $G$.

Lemma 2. If $M^{\prime}$ is a stable matching in $G^{\prime}$ then the projection of $M^{\prime}$ is a strongly dominant matching in $G$.

Proof. Let $M$ be the projection of $M^{\prime}$. In order to show that $M$ is a strongly dominant matching in $G$, we will construct a witness $(\boldsymbol{\alpha}, \mathbf{0})$ as given in Theorem 5. That is, we will construct $\boldsymbol{\alpha}$ as shown below such that $(\boldsymbol{\alpha}, \mathbf{0})$ is a feasible solution to LP3 and $\sum_{u \in V} \alpha_{u}=0$.

Set $\alpha_{u}=0$ for all vertices $u$ left unmatched in $M$. For each vertex $u$ matched in $M$ :

- if $\left(u^{+}, *\right) \in M^{\prime}$ then set $\alpha_{u}=1$; else set $\alpha_{u}=-1$.

Note that $\sum_{u \in V} \alpha_{u}=0$ since for each edge $(a, b) \in M$, we have $\alpha_{a}+\alpha_{b}=0$ and for each vertex $u$ that is unmatched in $M$, we have $\alpha_{u}=0$. We also have $\alpha_{u} \geq \operatorname{cost}_{M}(u, u)$ for all $u \in V$ since (i) $\alpha_{u}=0=\operatorname{cost}_{M}(u, u)$ for all $u$ left unmatched in $M$ and (ii) $\alpha_{u} \geq-1=\operatorname{cost}_{M}(u, u)$ for all $u$ matched in $M$.

We will now show that for every $(a, b) \in E, \alpha_{a}+\alpha_{b} \geq \operatorname{cost}_{M}(a, b)$. Recall that $\operatorname{cost}_{M}(a, b)$ is the sum of votes of $a$ and $b$ for each other over their respective assignments in $M$.

1. Suppose $\left(a^{+}, *\right) \in M^{\prime}$. So $\alpha_{a}=1$. We will consider 3 subcases here.

- The first subcase is that $\left(b^{+}, *\right) \in M^{\prime}$. So $\alpha_{b}=1$. Since $\operatorname{cost}_{M}(a, b) \leq 2$, it follows that $\alpha_{a}+\alpha_{b}=2 \geq \operatorname{cost}_{M}(a, b)$.
- The second subcase is that $\left(b^{-}, *\right) \in M^{\prime}$. So $\alpha_{b}=-1$. If $\left(a^{+}, b^{-}\right) \in M^{\prime}$ then $\operatorname{cost}_{M}(a, b)=$ $0=\alpha_{a}+\alpha_{b}$. So assume $\left(a^{+}, c^{-}\right)$and $\left(b^{-}, d^{+}\right)$belong to $M^{\prime}$. Since $M^{\prime}$ is stable, the edge $\left(a^{+}, b^{-}\right)$does not block $M^{\prime}$. Thus either (i) $a$ prefers $c^{-}$to $b^{-}$or (ii) $b$ prefers $d^{+}$to $a^{+}$. Hence $\operatorname{cost}_{M}(a, b) \leq 0$ and so $\alpha_{a}+\alpha_{b}=0 \geq \operatorname{cost}_{M}(a, b)$.
- The third subcase is that $b$ is unmatched in $M$. So $\alpha_{b}=0$. Since $M^{\prime}$ is stable, the edge $\left(a^{+}, b^{-}\right)$does not block $M^{\prime}$. Thus $a$ prefers its partner $c^{-}$in $M^{\prime}$ to $b^{-}$and so $\operatorname{cost}_{M}(a, b)=$ $0<\alpha_{a}+\alpha_{b}$.

2. Suppose $\left(a^{-}, *\right) \in M$. There are 3 subcases here as before. The case where $\left(b^{+}, *\right) \in M$ is totally analogous to the case where $\left(a^{+}, *\right)$ and $\left(b^{-}, *\right)$ are in $M$. So we will consider the remaining 2 subcases here.

- The first subcase is that $\left(b^{-}, *\right) \in M^{\prime}$. So $\alpha_{b}=-1$. Let $\left(a^{-}, c^{+}\right)$and ( $b^{-}, d^{+}$) belong to $M^{\prime}$. Since $M^{\prime}$ is stable, the edge $\left(a^{+}, b^{-}\right)$does not block $M^{\prime}$. So $b$ prefers $d^{+}$to $a^{+}$. Similarly, the edge $\left(a^{-}, b^{+}\right)$does not block $M^{\prime}$. Hence $a$ prefers $c^{+}$to $b^{+}$. Thus both a and $b$ prefer their respective partners in $M$ to each other, i.e., $\operatorname{cost}_{M}(a, b)=-2$. So we have $\alpha_{a}+\alpha_{b}=-2=\operatorname{cost}_{M}(a, b)$.
- The second subcase is that $b$ is unmatched in $M$. Then the edge $\left(a^{+}, b^{-}\right)$blocks $M^{\prime}$ since $a$ prefers $b^{-}$to $c^{+}$(for any neighbor $c$ ) and $b$ prefers to be matched to $a^{+}$than be left unmatched. Since $M^{\prime}$ is stable and has no blocking edge, this means that this subcase does not arise.

3. Suppose $a$ is unmatched in $M$. Then $\left(b^{+}, *\right) \in M^{\prime}$ (otherwise $\left(a^{-}, b^{+}\right)$blocks $M^{\prime}$ ); moreover, $b$ prefers its partner $d^{-}$in $M^{\prime}$ to $a^{-}$. So we have $\operatorname{cost}_{M}(a, b)=0<\alpha_{a}+\alpha_{b}$.

Thus we always have $\alpha_{a}+\alpha_{b} \geq \operatorname{cost}_{M}(a, b)$ and hence $(\boldsymbol{\alpha}, \mathbf{0})$ is a valid witness of $M$. Since $\boldsymbol{\alpha}$ satisfies the conditions in Theorem 5, $M$ is a strongly dominant matching in $G$.

We will now show that if $G^{\prime}$ has no stable matching, then $G$ has no strongly dominant matching.

## Lemma 3. If $G$ admits a strongly dominant matching then $G^{\prime}$ admits a stable matching.

Proof. Let $M$ be a strongly dominant matching in $G=(V, E)$. Let $(\boldsymbol{\alpha}, \mathbf{0})$ be a witness of $M$ as given in Theorem 5. That is, $\alpha_{u}=0$ for $u$ unmatched in $M$ and $\alpha_{u}= \pm 1$ for $u$ matched in $M$. Moreover, for each $(u, v) \in M, \alpha_{u}+\alpha_{v}=\operatorname{cost}_{M}(u, v)=0$ by complementary slackness on LP3, so $\left(\alpha_{u}, \alpha_{v}\right)$ is either $(1,-1)$ or $(-1,1)$.

We will construct a stable matching $M^{\prime}$ in $G^{\prime}$ as follows. For each $(u, v) \in M$ :

- if $\left(\alpha_{u}, \alpha_{v}\right)=(1,-1)$ then add $\left(u^{+}, v^{-}\right)$to $M^{\prime} ;$ else add $\left(u^{-}, v^{+}\right)$to $M^{\prime}$.

We will show that no edge in $E^{\prime} \backslash M^{\prime}$ blocks $M^{\prime}$. Let $\left(a^{+}, b^{-}\right) \notin M^{\prime}$. We consider the following cases here:
Case 1. Suppose $\alpha_{b}=1$. Then $\left(b^{+}, d^{-}\right) \in M^{\prime}$ where $d=M(b)$. Since $b$ prefers $d^{-}$to $a^{+},\left(a^{+}, b^{-}\right)$ is not a blocking edge to $M^{\prime}$.
Case 2. Suppose $\alpha_{b}=-1$. Then $\left(b^{-}, d^{+}\right) \in M^{\prime}$ where $d=M(b)$. We have 2 sub-cases here: (i) $\alpha_{a}=1$ and (ii) $\alpha_{a}=-1$. Note that $\alpha_{a} \neq 0$ as the edge $(a, b)$ would not be covered by $\alpha_{a}+\alpha_{b}$ then. This is because if $\alpha_{a}=0$ then $a$ is unmatched in $M$ and $\operatorname{cost}_{M}(a, b)=0$ while $\alpha_{a}+\alpha_{b}=-1$.

- In sub-case (i), some edge $\left(a^{+}, c^{-}\right)$belongs to $M^{\prime}$. We know that $\operatorname{cost}_{M}(a, b) \leq \alpha_{a}+\alpha_{b}=0$, so either (1) $a$ prefers $M(a)=c$ to $b$ or (2) $b$ prefers $M(b)=d$ to $a$. Hence either (1) $a$ prefers $c^{-}$ to $b^{-}$or (2) $b$ prefers $d^{+}$to $a^{+}$. Thus $\left(a^{+}, b^{-}\right)$is not a blocking edge to $M^{\prime}$.
- In sub-case (ii), some edge $\left(a^{-}, c^{+}\right)$belongs to $M^{\prime}$. We know that $\operatorname{cost}_{M}(a, b) \leq \alpha_{a}+\alpha_{b}=-2$, so $a$ prefers $M(a)=c$ to $b$ and $b$ prefers $M(b)=d$ to $a$. Thus $b$ prefers $d^{+}$to $a^{+}$, hence $\left(a^{+}, b^{-}\right)$ is not a blocking edge to $M^{\prime}$.

Case 3. Suppose $\alpha_{b}=0$. Thus $b$ was unmatched in $M$. Each of $b$ 's neighbors has to be matched in $M$ to a neighbor that it prefers to $b$, otherwise $M$ would be unpopular. We have $\alpha_{a}+\alpha_{b} \geq$ $\operatorname{cost}_{M}(a, b)=0$, hence it follows that $\alpha_{a}=1$. Thus $\left(a^{+}, c^{-}\right) \in M^{\prime}$ where $c$ is a neighbor that $a$ prefers to $b$. So $\left(a^{+}, b^{-}\right)$is not a blocking edge to $M^{\prime}$.

Lemmas 2 and 3 show that a strongly dominant matching is present in $G$ if and only if a stable matching is present in $G^{\prime}$. This finishes the proof of correctness of our algorithm. Since Irving's stable matching algorithm in $G^{\prime}$ can be implemented to run in linear time [19], we can conclude Theorem 2 stated in Section 1

## 4 The max-size popular matching problem in a roommates instance

In this section we prove the NP-hardness of the max-size popular matching problem in a roommates instance. We will show a reduction from the vertex cover problem.

Let $H=\left(V_{H}, E_{H}\right)$ be an instance of the vertex cover problem and let $V_{H}=\left\{1, \ldots, n_{H}\right\}$, i.e., $V_{H}=\left[n_{H}\right]$. We will build a roommates instance $G$ as follows: (see Fig. 1 )

- corresponding to every vertex $i \in V_{H}$, there will be 4 vertices $a_{i}, b_{i}, c_{i}, d_{i}$ in $G$ and
- corresponding to every edge $e=(i, j) \in E_{H}$, there will be 2 vertices $u_{i}^{e}$ and $u_{j}^{e}$ in $G$.


Fig. 1. The graph $G$ restricted to the adjacent vertices $i$ and $j$ in $H$ : the vertices $a_{t}, b_{t}, c_{t}, d_{t}$ in $G$ correspond to vertex $t \in\{i, j\}$ in $H$ and the vertices $u_{i}^{e}$ and $u_{j}^{e}$ in $G$ correspond to the edge $e=(i, j)$ in $H$. Vertex preferences in $G$ are indicated on the edges.

The preferences of the vertices $a_{i}, b_{i}, c_{i}, d_{i}$ are as follows, where $e_{1}, \ldots, e_{k}$ are all the edges in $H$ with vertex $i$ as an endpoint.

$$
a_{i}: b_{i} \succ c_{i} \succ d_{i} \quad b_{i}: a_{i} \succ u_{i}^{e_{1}} \cdots \succ u_{i}^{e_{k}} \succ c_{i} \quad c_{i}: a_{i} \succ b_{i} \quad d_{i}: a_{i} .
$$

The order among the vertices $u_{i}^{e_{1}}, \ldots, u_{i}^{e_{k}}$ in the preference list of $b_{i}$ is arbitrary. The preference list of vertex $u_{i}^{e}$ is $u_{j}^{e} \succ b_{i}$, where $e=(i, j)$ (see Fig. 1].

Observe that $G$ admits a stable matching $S=\left\{\left(a_{i}, b_{i}\right): 1 \leq i \leq n_{H}\right\} \cup\left\{\left(u_{i}^{e}, u_{j}^{e}\right): e=(i, j) \in E_{H}\right\}$. This is the set of blue edges in Fig. 1

Lemma 4. Let $M$ be a popular matching in $G$.

- For any $i \in\left[n_{H}\right]$, either $\left(a_{i}, b_{i}\right) \in M$ or $\left\{\left(a_{i}, d_{i}\right),\left(b_{i}, c_{i}\right)\right\} \subseteq M$.
- The set $C=\left\{i \in\left[n_{H}\right]:\left(a_{i}, b_{i}\right) \in M\right\}$ is a vertex cover of $H$.

Proof. The vertex $a_{i}$ is the top choice neighbor of all its neighbors $b_{i}, c_{i}, d_{i}$. Thus $a_{i}$ has to be matched in the popular matching $M$.

1. If $a_{i}$ is matched to $b_{i}$ then $\left(a_{i}, b_{i}\right) \in M$.
2. If $\left(a_{i}, d_{i}\right) \in M$, then $c_{i}$ also has to be matched - otherwise we get a more popular matching by replacing the edge $\left(a_{i}, d_{i}\right)$ with $\left(a_{i}, c_{i}\right)$. Since $c_{i}$ has degree 2, it has to be the case that $\left\{\left(a_{i}, d_{i}\right),\left(b_{i}, c_{i}\right)\right\} \subseteq M$.
3. Suppose $\left(a_{i}, c_{i}\right) \in M$. Since $a_{i}$ prefers $b_{i}$ to $c_{i}$, this means that $b_{i}$ has to be matched in $M$. Other than $a_{i}$ and $c_{i}$ (which are matched to each other), $b_{i}$ 's neighbors are $u_{i}^{e}$ for all edges $e$ incident to $i$ in $H$. So $\left(b_{i}, u_{i}^{e}\right) \in M$ for some $e=(i, j) \in E_{H}$. We will now construct a new matching $M^{\prime}$ as follows:

- replace the edges $\left(a_{i}, c_{i}\right),\left(b_{i}, u_{i}^{e}\right),\left(u_{j}^{e}, M\left(u_{j}^{e}\right)\right)$ in $M$ with the edges $\left(a_{i}, b_{i}\right)$ and $\left(u_{i}^{e}, u_{j}^{e}\right)$. So $c_{i}$ and $M\left(u_{j}^{e}\right)$ are unmatched in $M^{\prime}$, hence these 2 vertices prefer $M$ to $M^{\prime}$; however the 4 vertices $a_{i}, b_{i}, u_{i}^{e}, u_{j}^{e}$ prefer $M^{\prime}$ to $M$. Thus $M^{\prime}$ is more popular than $M$, a contradiction to the popularity of $M$. Hence $\left(a_{i}, c_{i}\right) \notin M$.

We now show the second part of the lemma. Let $(i, j) \in E_{H}$. We need to show that either $\left(a_{i}, b_{i}\right) \in M$ or $\left(a_{j}, b_{j}\right) \in M$. Suppose not. Then by the first part of this lemma, $\left(a_{i}, d_{i}\right),\left(b_{i}, c_{i}\right)$ are in $M$ and similarly, $\left(a_{j}, d_{j}\right),\left(b_{j}, c_{j}\right)$ are in $M$. Also $\left(u_{i}^{e}, u_{j}^{e}\right) \in M$.

Consider the matching $M^{\prime}$ obtained by replacing the 5 edges $\left(a_{i}, d_{i}\right),\left(b_{i}, c_{i}\right),\left(a_{j}, d_{j}\right),\left(b_{j}, c_{j}\right)$, and $\left(u_{i}^{e}, u_{j}^{e}\right)$ in $M$ with the 4 edges $\left(a_{i}, c_{i}\right),\left(b_{i}, u_{i}^{e}\right),\left(a_{j}, c_{j}\right)$, and $\left(b_{j}, u_{j}^{e}\right)$ (see Fig. 11). Among the 10 vertices involved here, the 6 vertices $a_{i}, b_{i}, c_{i}$ and $a_{j}, b_{j}, c_{j}$ prefer $M^{\prime}$ to $M$ while the 4 vertices $d_{i}, u_{i}^{e}$ and $d_{j}, u_{j}^{e}$ prefer $M$ to $M^{\prime}$. Thus $M^{\prime}$ is more popular than $M$, a contradiction to the popularity of M.

Hence for each edge $(i, j) \in E_{H}$ either $\left(a_{i}, b_{i}\right) \in M$ or $\left(a_{j}, b_{j}\right) \in M$. In other words, the set $U=\left\{i \in\left[n_{H}\right]:\left(a_{i}, b_{i}\right) \in M\right\}$ is a vertex cover of $H$.

Theorem 6. For any $1 \leq k \leq n_{H}$, the graph $H=\left(V_{H}, E_{H}\right)$ admits a vertex cover of size $k$ if and only if $G$ has a popular matching of size at least $m_{H}+2 n_{H}-k$, where $\left|E_{H}\right|=m_{H}$.

Proof. Suppose $H=\left(V_{H}, E_{H}\right)$ admits a vertex cover $U$ of size $k$. We will now build a popular matching $M$ in $G$ of size $m_{H}+2 n_{H}-k$.

- Add all edges $\left(u_{i}^{e}, u_{j}^{e}\right)$ in $G$ to $M$.
- For every $i \in U$, add the edge $\left(a_{i}, b_{i}\right)$ to $M$.
- For every $i \notin U$, add the edges $\left(a_{i}, d_{i}\right),\left(b_{i}, c_{i}\right)$ to $M$.

The size of $M$ is $m_{H}+|U|+2\left(n_{H}-|U|\right)=m_{H}+2 n_{H}-k$. We will prove $M$ is popular by showing a witness ( $\boldsymbol{\alpha}, \boldsymbol{z}$ ) for it. To begin with, initialize $z_{S}=0$ for all $S \in \Omega$.

- For every $i \in U$ : set $\alpha_{a_{i}}=\alpha_{b_{i}}=\alpha_{c_{i}}=\alpha_{d_{i}}=0$.
- For every $i \notin U$ : set $\alpha_{a_{i}}=1$ and $\alpha_{b_{i}}=\alpha_{c_{i}}=\alpha_{d_{i}}=-1$; also set $z_{S_{i}}=2$ where $S_{i}=\left\{a_{i}, b_{i}, c_{i}\right\}$.
- For every edge $e=(i, j) \in E_{H}$ : if $i \in U$ then set $\alpha_{u_{i}^{e}}=-1$ and $\alpha_{u_{j}^{e}}=1$; else set $\alpha_{u_{i}^{e}}=1$ and $\alpha_{u_{j}^{e}}=-1$.
It is easy to check that the above setting of $(\boldsymbol{\alpha}, \boldsymbol{z})$ covers all edges of $G$. In particular, for $i \in U$, we have $\operatorname{cost}_{M}\left(b_{i}, u_{i}^{e}\right)=-1-1=-2$ while $\alpha_{b_{i}}=0$ and $\alpha_{u_{i}^{e}}=-1$, thus $\alpha_{b_{i}}+\alpha_{u_{i}^{e}} \geq \operatorname{cost}_{M}\left(b_{i}, u_{i}^{e}\right)$. Similarly, when $i \notin U, \operatorname{cost}_{M}\left(b_{i}, u_{i}^{e}\right)=1-1=0$ and we have $\alpha_{b_{i}}=-1$ and $\alpha_{u_{i}^{e}}=1$ here. Moreover,

$$
\begin{equation*}
\sum_{v \in V} \alpha_{v}+\sum_{S \in \Omega}\lfloor|S| / 2\rfloor \cdot z_{S}=\sum_{i \notin U}-2+\sum_{i \notin U} 2=0 . \tag{1}
\end{equation*}
$$

This is because $\alpha_{v}=0$ for all vertices $v$ unmatched in $M$ and $\alpha_{u}+\alpha_{v}=0$ for all edges $(u, v) \in M$ except the edges $\left(b_{i}, c_{i}\right)$ where $i \notin U$. For each $i \notin U$, we have $\alpha_{b_{i}}+\alpha_{c_{i}}=-2$ and we also have $z_{S_{i}}=2$ where $S_{i}=\left\{a_{i}, b_{i}, c_{i}\right\}$. The sum in Equation (1) is 0 and so $M$ is a popular matching. Hence $G$ has a popular matching of size $m_{H}+2 n_{H}-k$.

We will now show the converse. Let $M$ be a popular matching in $G$ of size at least $m_{H}+2 n_{H}-k$. We know that $U=\left\{i \in\left[n_{H}\right]:\left(a_{i}, b_{i}\right) \in M\right\}$ is a vertex cover of $H$ (by Lemma 4). We will show that $|U| \leq k$.

It follows from Lemma 4 that all edges $\left(u_{i}^{e}, u_{j}^{e}\right)$ belong to any popular matching. So these account for $m_{H}$ many edges in $M$. We also know that for every $i \in V_{H}$, either $\left(a_{i}, b_{i}\right) \in M$ or $\left\{\left(a_{i}, d_{i}\right),\left(b_{i}, c_{i}\right)\right\} \subseteq M$ (by Lemma 4 . Thus $|M|=m_{H}+|U|+2\left(n_{H}-|U|\right)$. Since $|M| \geq m_{H}+2 n_{H}-k$, it follows that $|U| \leq k$. Thus the graph $G$ has a vertex cover of size $k$.

We can now conclude Theorem 1 stated in Section 1 .
Remark. The rural hospitals theorem [24] for stable matchings in a roommates instance $G$ says that every stable matching in $G$ matches the same subset of vertices. Such a statement is not true for max-size popular matchings in a roommates instance as seen in the instance on 10 vertices in Fig. 1. This instance has two max-size popular matchings (these are of size 4): one leaves $c_{i}$ and $d_{i}$ unmatched while another leaves $c_{j}$ and $d_{j}$ unmatched.

## 5 Max-weight popular matchings in bipartite instances

In this section we will prove the NP-hardness of the max-weight popular matching problem in a bipartite instance $G=(A \cup B, E)$ on $n$ vertices and with edge weights. Call any $e \in E$ a popular edge if there is a popular matching $N$ in $G$ such that $e \in N$.

Let $M$ be a popular matching in $G=(A \cup B, E)$. Observe that for any popular matching $N$ in $G$, the perfect matching $\tilde{N}$ in $\tilde{G}$ is an optimal solution of LP1 (see Section 2) and a witness $\boldsymbol{\alpha}$ of $M$ is an optimal solution of LP2. Lemma 5 follows from complementary slackness conditions on LP1.

Lemma 5. Let $M$ be a popular matching in $G$ and let $\boldsymbol{\alpha} \in\{0, \pm 1\}^{n}$ be a witness of $M$.

1. For any popular edge $(a, b) \in E$, the parities of $\alpha_{a}$ and $\alpha_{b}$ have to be the same.
2. If $u$ is a vertex in $G$ that is left unmatched in a stable matching in $G$ (call $u$ unstable) then $\alpha_{u}=\operatorname{cost}_{M}(u, u)$; thus $\alpha_{u}=0$ if $u$ is left unmatched in $M$, otherwise $\alpha_{u}=-1$.

Proof. Let $(a, b)$ be a popular edge. So $(a, b) \in N$ for some popular matching $N$. Since $N$ is popular, $\Delta(N, M)=0$ and the perfect matching $\tilde{N}$ is an optimal solution to the max-cost perfect matching LP in the graph $\tilde{G}$ with cost function $\operatorname{cost}_{M}$ (see LP1 from Section 2). Since $\boldsymbol{\alpha}$ is an optimal solution to the dual LP (see LP2), it follows from complementary slackness that $\alpha_{a}+\alpha_{b}=\operatorname{cost}_{M}(a, b)$. Observe that $\operatorname{cost}_{M}(a, b) \in\{ \pm 2,0\}$ (an even number). Hence the integers $\alpha_{a}$ and $\alpha_{b}$ have the same parity. This proves part 1.

Part 2 also follows from complementary slackness. Let $S$ be a stable matching in $G$ and let $u$ be a vertex left unmatched in $S$. So the perfect matching $\tilde{S}$ contains the edge $(u, u)$. Since $\tilde{S}$ is an optimal solution to LP1, we have $\alpha_{u}=\operatorname{cost}_{M}(u, u)$ by complementary slackness. Thus when $u$ is left unmatched in $M, \alpha_{u}=0$, else $\alpha_{u}=-1$.

The popular subgraph. We will define a subgraph $F_{G}=\left(A \cup B, E_{F}\right)$ called the popular subgraph of $G$, where $E_{F}$ is the set of popular edges in $E$. The subgraph $F_{G}$ need not be connected: let $\mathcal{C}=\left\{\mathcal{C}_{1}, \ldots, \mathcal{C}_{h}\right\}$ be the set of connected components in $F_{G}$.

Each component $\mathcal{C}_{j}$ that is a singleton set consists of a single unpopular vertex, i.e., one left unmatched in all popular matchings. Every non-singleton component $\mathcal{C}_{i}$ has an even number of vertices. This is because a max-size popular matching matches all vertices except the ones in singleton sets in $\mathcal{C}$ and vertices in $\mathcal{C}_{i}$ are matched to each other (15).

It is also known that all stable vertices (those matched in stable matchings in $G$ ) have to be matched in every popular matching [16]. Let $M$ be any popular matching in $G$ and let $\boldsymbol{\alpha} \in\{0, \pm 1\}^{n}$ be a witness of $M$ 's popularity. The following lemma will be useful to us.

Lemma 6. For any connected component $\mathcal{C}_{i}$ in $F_{G}$, either $\alpha_{u}=0$ for all vertices $u \in \mathcal{C}_{i}$ or $\alpha_{u}= \pm 1$ for all vertices $u \in \mathcal{C}_{i}$. Moreover, if $\mathcal{C}_{i}$ contains one or more unstable vertices, either all the unstable vertices in $\mathcal{C}_{i}$ are matched in $M$ or none of them is matched in $M$.

Proof. Let $u$ and $v$ be any 2 vertices in $\mathcal{C}_{i}$. Since $u, v$ are in the same connected component in $F_{G}$, there is a $u-v$ path $\rho$ in $G$ such that every edge in $\rho$ is a popular edge. The endpoints of each popular edge have the same parity in $\boldsymbol{\alpha}$ (by part 1 of Lemma 5), hence $\alpha_{u}$ and $\alpha_{v}$ have the same parity. Thus either $\alpha_{u}=\alpha_{v}=0$ or both $\alpha_{u}, \alpha_{v} \in\{ \pm 1\}$.

Let $\mathcal{C}_{i}$ be a connected component with one or more unstable vertices, i.e., those left unmatched in a stable matching. Since all $\alpha$-values in $\mathcal{C}_{i}$ have the same parity, either (i) all vertices $v$ in $\mathcal{C}_{i}$ satisfy $\alpha_{v}=0$ or (ii) all vertices $v$ in $\mathcal{C}_{i}$ satisfy $\alpha_{v}= \pm 1$. For any unstable vertex $u$, we have $\alpha_{u}=\operatorname{cost}_{M}(u, u)$ (by part 2 of Lemma 5), hence in case (i), all unstable vertices in $\mathcal{C}_{i}$ are left unmatched in $M$ and in case (ii), all unstable vertices in $\mathcal{C}_{i}$ are matched in $M$.

The NP-hardness reduction. Given a graph $H=\left(V_{H}, E_{H}\right)$ which is an instance of the vertex cover problem, we will now build an instance $G=(A \cup B, E)$ with strict preference lists such that the vertex cover problem in $H$ reduces to the max-weight popular matching problem in $G$. Let $V_{H}=\left\{1, \ldots, n_{H}\right\}$.

- For every edge $e \in E_{H}$, there will be a gadget $D_{e}$ in $G$ on 6 vertices $s_{e}, t_{e}, s_{e}^{\prime}, t_{e}^{\prime}, s_{e}^{\prime \prime}, t_{e}^{\prime \prime}$.
- For every vertex $i \in V_{H}$, there will be a gadget $C_{i}$ in $G$ on 4 vertices $a_{i}, b_{i}, a_{i}^{\prime}, b_{i}^{\prime}$.
- There are 2 more vertices in $G$ : these are $a_{0}$ and $b_{0}$.

Thus $A=\left\{a_{0}\right\} \cup\left\{a_{i}, a_{i}^{\prime}: i \in V_{H}\right\} \cup\left\{s_{e}, s_{e}^{\prime}, s_{e}^{\prime \prime}: e \in E_{H}\right\}$ and $B=\left\{b_{0}\right\} \cup\left\{b_{i}, b_{i}^{\prime}: i \in V_{H}\right\} \cup$ $\left\{t_{e}, t_{e}^{\prime}, t_{e}^{\prime \prime}: e \in E_{H}\right\}$. We now describe the edge set of $G$. We first describe the preference lists of the 6 vertices in $D_{e}$, where $e=(i, j)$ and $i<j$ (see Fig. 2).

$$
\begin{array}{lll}
s_{e}^{\prime}: t_{e}^{\prime} \succ t_{e} & s_{e}^{\prime \prime}: t_{e}^{\prime \prime} \succ t_{e} & s_{e}: t_{e}^{\prime} \succ b_{j} \succ t_{e}^{\prime \prime} \\
t_{e}^{\prime}: s_{e}^{\prime} \succ s_{e} & t_{e}^{\prime \prime}: s_{e}^{\prime \prime} \succ s_{e} & t_{e}: s_{e}^{\prime \prime} \succ a_{i} \succ s_{e}^{\prime}
\end{array}
$$

Here $s_{e}^{\prime}$ and $t_{e}^{\prime}$ are each other's top choices and similarly, $s_{e}^{\prime \prime}$ and $t_{e}^{\prime \prime}$ are each other's top choices. The vertex $s_{e}$ 's top choice is $t_{e}^{\prime}$, second choice is $b_{j}$, and third choice is $t_{e}^{\prime \prime}$. For $t_{e}$, the order is $s_{e}^{\prime \prime}$, followed by $a_{i}$, and then $s_{e}^{\prime}$. Recall that $e=(i, j)$ and $i<j$.


Fig. 2. To the left is the gadget $C_{i}$ on vertices $a_{i}, b_{i}, a_{i}^{\prime}, b_{i}^{\prime}$ and to the right is the gadget $D_{e}$ on vertices $s_{e}, t_{e}, s_{e}^{\prime}, t_{e}^{\prime}, s_{e}^{\prime \prime}, t_{e}^{\prime \prime}$; the vertices $a_{0}$ and $b_{0}$ are adjacent to $b_{i}$ and $a_{i}$, respectively.

We now describe the preference lists of the 4 vertices $a_{i}, b_{i}, a_{i}^{\prime}, b_{i}^{\prime}$ in $C_{i}$ (see Fig. 22).

$$
\begin{aligned}
a_{i}^{\prime}: b_{i} \succ b_{i}^{\prime} & a_{i}: b_{i} \succ b_{i}^{\prime} \succ b_{0} \succ \cdots \\
b_{i}^{\prime}: a_{i} \succ a_{i}^{\prime} & b_{i}: a_{i} \succ a_{i}^{\prime} \succ a_{0} \succ \cdots
\end{aligned}
$$

- Both $a_{i}^{\prime}$ and $a_{i}$ have $b_{i}$ as their top choice and $b_{i}^{\prime}$ as their second choice. Similarly, both $b_{i}^{\prime}$ and $b_{i}$ have $a_{i}$ as their top choice and $a_{i}^{\prime}$ as their second choice.
- The vertex $a_{i}$ has other neighbors: its third choice is $b_{0}$ followed by all the vertices $t_{e_{1}}, t_{e_{2}}, \ldots$ where $i$ is the lower-indexed endpoint of $e_{1}, e_{2}, \ldots$ Similarly, $b_{i}$ has $a_{0}$ as its third choice followed by all the vertices $s_{e_{1}^{\prime}}, s_{e_{2}^{\prime}}, \ldots$ where $i$ is the higher-indexed endpoint of $e_{1}^{\prime}, e_{2}^{\prime}, \ldots$
- The order among the vertices $t_{e_{1}}, t_{e_{2}}, \ldots$ (similarly, $s_{e_{1}^{\prime}}, s_{e_{2}^{\prime}}, \ldots$ ) in the preference list of $a_{i}$ (resp., $b_{i}$ ) does not matter; hence these are represented as $\cdots$ in these preference lists.

The vertex $a_{0}$ has $b_{1}, \ldots, b_{n_{H}}$ as its neighbors and its preference list is some arbitrary permutation of these vertices. Similarly, the vertex $b_{0}$ has $a_{1}, \ldots, a_{n_{H}}$ as its neighbors and its preference list is some arbitrary permutation of these vertices.

Lemma 7. No popular matching in $G$ matches either $a_{0}$ or $b_{0}$.
Proof. We will first show that $M=\left\{\left(a_{i}, b_{i}^{\prime}\right),\left(a_{i}^{\prime}, b_{i}\right): i \in V_{H}\right\} \cup\left\{\left(s_{e}^{\prime}, t_{e}^{\prime}\right),\left(s_{e}^{\prime \prime}, t_{e}^{\prime \prime}\right): e \in E_{H}\right\}$ is a popular matching in $G$. We prove the popularity of $M$ by showing a vector $\boldsymbol{\alpha} \in\{0, \pm 1\}^{n}$ that will be a witness to M's popularity.

- For vertices in $C_{i}$, where $i \in V_{H}$ : let $\alpha_{a_{i}}=\alpha_{b_{i}}=1, \alpha_{a_{i}^{\prime}}=\alpha_{b_{i}^{\prime}}=-1$.
- For any vertex $u$ in $D_{e}$, where $e \in E_{H}$ : let $\alpha_{u}=0$.
- Let $\alpha_{a_{0}}=\alpha_{b_{0}}=0$.

It can be checked that $\alpha_{u}+\alpha_{v} \geq \operatorname{cost}_{M}(u, v)$ for all edges $(u, v)$ in $G$. Also, $\alpha_{u} \geq \operatorname{cost}_{M}(u, u)$ for all $u \in A \cup B$. Since $\alpha_{u}=0$ for all vertices $u$ unmatched in $M$ and $\alpha_{u}+\alpha_{v}=0$ for every edge $(u, v)$ in $M, \sum_{u \in A \cup B} \alpha_{u}=0$.

Suppose $a_{0}$ is matched in some popular matching $N$, i.e., $\left(a_{0}, b_{i}\right) \in N$ for some $i \in\left[n_{H}\right]$. Then $\left(a_{0}, b_{i}\right)$ is a popular edge. So it follows from part 1 of Lemma 5 that the parities of $\beta_{a_{0}}$ and $\beta_{b_{i}}$ have to be the same, for any witness $\boldsymbol{\beta}$ of the popular matching $M$. However $\alpha_{a_{0}}=0$ while $\alpha_{b_{i}}=1$ for the witness $\boldsymbol{\alpha}$ described above. This is a contradiction to ( $a_{0}, b_{i}$ ) being a popular edge and hence no popular matching in $G$ matches $a_{0}$ (similarly, $b_{0}$ ).

Lemma 8. Let $N$ be any popular matching in $G$. For each $i \in\left\{1, \ldots, n_{H}\right\}$ : either $\left(a_{i}, b_{i}\right)$ and $\left(a_{i}^{\prime}, b_{i}^{\prime}\right)$ are in $N$ or $\left(a_{i}, b_{i}^{\prime}\right)$ and $\left(a_{i}^{\prime}, b_{i}\right)$ are in $N$.

Proof. We know from Lemma 7 that in the popular subgraph $F_{G}$, the vertices $a_{0}$ and $b_{0}$ are singleton sets. We now claim that each $C_{i}$ forms a maximal connected component in $F_{G}$. This implies that $a_{i}$ has only 2 possible partners in any popular matching $N$ : either $b_{i}$ or $b_{i}^{\prime}$. Thus either (1) $\left\{\left(a_{i}, b_{i}\right),\left(a_{i}^{\prime}, b_{i}^{\prime}\right)\right\} \subset N$ or $(2)\left\{\left(a_{i}, b_{i}^{\prime}\right),\left(a_{i}^{\prime}, b_{i}\right)\right\} \subset N$.

It is easy to see that the 4 vertices of $C_{i}$ belong to the same connected component in $F_{G}$ : this is because all the 4 edges $\left(a_{i}, b_{i}\right),\left(a_{i}^{\prime}, b_{i}^{\prime}\right),\left(a_{i}, b_{i}^{\prime}\right),\left(a_{i}^{\prime}, b_{i}\right)$ are popular. The former 2 edges belong to the stable matching $S=\left\{\left(a_{i}, b_{i}\right),\left(a_{i}^{\prime}, b_{i}^{\prime}\right): i \in V_{H}\right\} \cup\left\{\left(s_{e}^{\prime}, t_{e}^{\prime}\right),\left(s_{e}^{\prime \prime}, t_{e}^{\prime \prime}\right): e \in E_{H}\right\}$ while the latter 2 edges belong to the popular matching $M$ defined in Lemma 7 .

Consider the matching $M$ and its witness $\boldsymbol{\alpha}$ defined in the proof of Lemma 7 the parities of $\alpha_{a_{i}}$ and $\alpha_{t_{e}}$ for any $i \in V_{H}$ and $e \in E_{H}$ are different. Hence it follows from part 1 of Lemma 5 that the edge $\left(a_{i}, t_{e}\right)$ is not a popular edge. Similarly, the edge $\left(s_{e}, b_{j}\right)$ is not a popular edge. Thus each $C_{i}$ forms a maximal connected component in $F_{G}$.

Lemma 8 will be important in our reduction. Let $\boldsymbol{\alpha}$ be any witness of $N$.

- In the first possibility of Lemma 8 i.e., when $\left(a_{i}, b_{i}\right),\left(a_{i}^{\prime}, b_{i}^{\prime}\right)$ are in $N$, we have $\alpha_{u}=0$ for all $u \in C_{i}$. This is because $a_{0}$ and $b_{0}$ are unmatched in $N$, so $\alpha_{a_{i}}+\alpha_{b_{0}} \geq \operatorname{cost}_{N}\left(a_{i}, b_{0}\right)=0$ and $\alpha_{a_{0}}+\alpha_{b_{i}} \geq \operatorname{cost}_{N}\left(a_{0}, b_{i}\right)=0$. Also $\alpha_{a_{0}}=\alpha_{b_{0}}=0$ (by Lemma 5). So $\alpha_{a_{i}} \geq 0$ and $\alpha_{b_{i}} \geq 0$. We also have $\alpha_{a_{i}}+\alpha_{b_{i}}=\operatorname{cost}_{N}\left(a_{i}, b_{i}\right)=0$. Thus $\alpha_{a_{i}}=\alpha_{b_{i}}=0$.
- In the second possibility of Lemma 8, i.e., when $\left(a_{i}, b_{i}^{\prime}\right),\left(a_{i}^{\prime}, b_{i}\right)$ are in $N$, we have $\alpha_{u}= \pm 1$ for all $u \in C_{i}$. This is because $\operatorname{cost}_{N}\left(a_{i}, b_{i}\right)=2$, thus $\alpha_{a_{i}}=\alpha_{b_{i}}=1$.

Edge weights. We now assign weights to the edges in $G$. For all $e \in E_{H}$, let $w\left(s_{e}, t_{e}^{\prime}\right)=w\left(s_{e}, t_{e}^{\prime \prime}\right)=$ $w\left(s_{e}^{\prime}, t_{e}\right)=w\left(s_{e}^{\prime \prime}, t_{e}\right)=2$, i.e., all the edges between $s_{e}, t_{e}$ and their neighbors in $D_{e}$ have weight 2 . For all $i \in\left[n_{H}\right]$, let $w\left(a_{i}, b_{i}\right)=w\left(a_{i}^{\prime}, b_{i}^{\prime}\right)=2$. Set $w(e)=1$ for all other edges $e$ in $G$.

Let $N$ be a max-weight popular matching in $G$ and let $\boldsymbol{\alpha} \in\{0, \pm 1\}^{n}$ be a witness of $N$ 's popularity. Let $U_{N}$ be the set of those indices $i \subseteq\left[n_{H}\right]$ such that $\alpha_{u}= \pm 1$ for all $u \in C_{i}$.

Lemma 9. The set $U_{N}$ is a vertex cover of the graph $H$.
Proof. Edge weights were assigned in $G$ such that the following claim (proof given below) holds for any max-weight popular matching $N$.

Claim 2 For every $e \in E_{H}$, both $s_{e}$ and $t_{e}$ have to be matched in $N$.
Consider any edge $e=(i, j) \in E_{H}$, let $i<j$. It follows from Claim 2 and Lemma 8 that either the pair $\left(s_{e}, t_{e}^{\prime}\right),\left(s_{e}^{\prime}, t_{e}\right)$ or the pair $\left(s_{e}, t_{e}^{\prime \prime}\right),\left(s_{e}^{\prime \prime}, t_{e}\right)$ is in $N$.

- If $\left(s_{e}, t_{e}^{\prime}\right)$ and $\left(s_{e}^{\prime}, t_{e}\right)$ are in $N$, then $\operatorname{cost}_{N}\left(a_{i}, t_{e}\right)=0$ since $t_{e}$ prefers $a_{i}$ to $s_{e}^{\prime}$ while $a_{i}$ prefers both $b_{i}$ and $b_{i}^{\prime}\left(\right.$ its possible partners in $N$ ) to $t_{e}$. It follows from part 2 of Lemma 5 that $\alpha_{t_{e}}=-1$, thus $\alpha_{a_{i}}$ has to be 1 so that $\alpha_{a_{i}}+\alpha_{t_{e}} \geq 0$. Recall that $\boldsymbol{\alpha}$ is a witness of $N$ 's popularity.
- If $\left(s_{e}, t_{e}^{\prime \prime}\right)$ and $\left(s_{e}^{\prime \prime}, t_{e}\right)$ are in $N$, then $\operatorname{cost}_{N}\left(s_{e}, b_{j}\right)=0$ since $s_{e}$ prefers $b_{j}$ to $t_{e}^{\prime \prime}$ while $b_{j}$ prefers both $a_{j}$ and $a_{j}^{\prime}$ (its possible partners in $N$ ) to $s_{e}$. It follows from part 2 of Lemma 5 that $\alpha_{s_{e}}=-1$, thus $\alpha_{b_{j}}$ has to be 1 so that $\alpha_{s_{e}}+\alpha_{b_{j}} \geq 0$.

Thus at least one of $C_{i}, C_{j}$ assigns the $\alpha$-values of its vertices to $\pm 1$. Hence for every edge $(i, j) \in E_{H}$, at least one of $i, j$ is in $U_{N}$, in other words, $U_{N}$ is a vertex cover of $H$.

Proof of Claim 2, Consider any $e \in E_{H}$. Since $s_{e}^{\prime}, t_{e}^{\prime}, s_{e}^{\prime \prime}$, and $t_{e}^{\prime \prime}$ are stable vertices in $G$, they have to be matched in every popular matching in $G$. Thus any popular matching in $G$ either matches both $s_{e}, t_{e}$ or neither $s_{e}$ nor $t_{e}$. Recall from the proof of Lemma 8 that there is no popular edge between $D_{e}$ and either $C_{i}$ or $C_{j}$.

Let $N$ be a max-weight popular matching in $G$. We now need to show that $N$ matches both $s_{e}$ and $t_{e}$. Suppose $N$ matches neither $s_{e}$ nor $t_{e}$. Thus $\left(s_{e}^{\prime}, t_{e}^{\prime}\right)$ and $\left(s_{e}^{\prime \prime}, t_{e}^{\prime \prime}\right)$ are in $N$. Hence the weight contributed by vertices in $D_{e}$ to $w(N)$ is 2 .

Let $e=(i, j)$ where $i<j$. We know from Lemma 8 that either $\left(a_{i}, b_{i}^{\prime}\right),\left(a_{i}^{\prime}, b_{i}\right)$ are in $N$ or $\left(a_{i}, b_{i}\right),\left(a_{i}^{\prime}, b_{i}^{\prime}\right)$ are in $N$.
Case 1: Suppose $\left\{\left(a_{i}, b_{i}^{\prime}\right),\left(a_{i}^{\prime}, b_{i}\right)\right\} \subset N$. This means $\operatorname{cost}_{N}\left(a_{i}, b_{i}\right)=2$ and hence $\alpha_{a_{i}}=\alpha_{b_{i}}=1$ in any witness $\boldsymbol{\alpha}$ of $N$.

Consider the matching $N^{\prime}=N \cup\left\{\left(s_{e}, t_{e}^{\prime}\right),\left(s_{e}^{\prime}, t_{e}\right)\right\} \backslash\left\{\left(s_{e}^{\prime}, t_{e}^{\prime}\right)\right\}$. That is, we replace the edge $\left(s_{e}^{\prime}, t_{e}^{\prime}\right)$ in $N$ with the edges $\left(s_{e}, t_{e}^{\prime}\right)$ and $\left(s_{e}^{\prime}, t_{e}\right)$. It is easy to prove that $N^{\prime}$ is also popular; we will show a witness $\boldsymbol{\beta}$ to prove the popularity of $N^{\prime}$. Let $\boldsymbol{\alpha}$ be a witness of $N$ and let $\beta_{u}=\alpha_{u}$ for all $u \in A \cup B$ except for the vertices in $D_{e}$. For the vertices in $D_{e}$, let $\beta_{s_{e}}=\beta_{t_{e}}=\beta_{t_{e}^{\prime \prime}}=-1$ and $\beta_{s_{e}^{\prime}}=\beta_{t_{e}^{\prime}}=\beta_{s_{e}^{\prime \prime}}=1$.

It can be checked that $\sum_{u \in A \cup B} \beta_{u}=0$ and $\beta_{u}+\beta_{v} \geq \operatorname{cost}_{N^{\prime}}(u, v)$ for all $(u, v) \in E$ and $\beta_{u} \geq \operatorname{cost}_{N^{\prime}}(u, u)$ for all $u \in A \cup B$. In particular, $\operatorname{cost}_{N^{\prime}}\left(a_{i}, t_{e}\right)=0=\beta_{a_{i}}+\beta_{t_{e}}$. Moreover, we have $w\left(N^{\prime}\right)=w(N)+3$. Thus there is a popular matching $N^{\prime}$ in $G$ with a larger weight than $N$, a contradiction to our assumption that $N$ is a max-weight popular matching in $G$.
Case 2: Suppose $\left\{\left(a_{i}, b_{i}\right),\left(a_{i}^{\prime}, b_{i}^{\prime}\right)\right\} \subset N$. Consider the matching $N^{\prime \prime}$ which is exactly the same as $N$ except that the edges $\left(s_{e}^{\prime}, t_{e}^{\prime}\right),\left(a_{i}, b_{i}\right)$ and $\left(a_{i}^{\prime}, b_{i}^{\prime}\right)$ are deleted from $N$ and the new edges are $\left(s_{e}, t_{e}^{\prime}\right),\left(s_{e}^{\prime}, t_{e}\right),\left(a_{i}, b_{i}^{\prime}\right)$, and $\left(a_{i}^{\prime}, b_{i}\right)$.

We claim that $N^{\prime \prime}$ is also popular and we prove this by showing a witness $\gamma$. Let $\gamma_{u}=\alpha_{u}$ for all $u \in A \cup B(\boldsymbol{\alpha}$ is a witness of $N)$ except for the vertices in $C_{i} \cup D_{e}$. For the vertices in $C_{i} \cup D_{e}$, let $\gamma_{s_{e}}=\gamma_{t_{e}}=\gamma_{t_{e}^{\prime \prime}}=-1, \gamma_{s_{e}^{\prime}}=\gamma_{t_{e}^{\prime}}=\gamma_{s_{e}^{\prime \prime}}=1$, and $\gamma_{a_{i}}=\gamma_{b_{i}}=1, \gamma_{a_{i}^{\prime}}=\gamma_{b_{i}^{\prime}}=-1$.

It can be checked that $\sum_{u \in A \cup B} \gamma_{u}=0$ and $\gamma_{u}+\gamma_{v} \geq \operatorname{cost}_{N^{\prime \prime}}(u, v)$ for all $(u, v) \in E$ and $\gamma_{u} \geq \operatorname{cost}_{N^{\prime \prime}}(u, u)$ for all $u \in A \cup B$. Thus $N^{\prime \prime}$ is a popular matching. Moreover, we have $w\left(N^{\prime \prime}\right)=$ $w(N)+3-2=w(N)+1$. Thus there is a popular matching $N^{\prime \prime}$ in $G$ with a larger weight than $N$, a contradiction again. Hence we can conclude that both $s_{e}$ and $t_{e}$ have to be matched in $N$.

Theorem 7. For any integer $1 \leq k \leq n_{H}$, the graph $H=\left(V_{H}, E_{H}\right)$ admits a vertex cover of size $k$ if and only if $G$ admits a popular matching of weight at least $5 m_{H}+4 n_{H}-2 k$, where $\left|E_{H}\right|=m_{H}$.

Proof. Let $U$ be a vertex cover of size $k$ in $H$. Using $U$, we will construct a matching $M$ in $G$ of weight $5 m_{H}+4 n_{H}-2 k$ and a witness $\boldsymbol{\alpha}$ to $M$ 's popularity as follows. For every $i \in\left[n_{H}\right]$ :

- if $i \in U$ then include edges $\left(a_{i}, b_{i}^{\prime}\right)$ and $\left(a_{i}^{\prime}, b_{i}\right)$ in $M$; set $\alpha_{a_{i}}=\alpha_{b_{i}}=1$ and set $\alpha_{a_{i}^{\prime}}=\alpha_{b_{i}^{\prime}}=-1$.
- else include edges $\left(a_{i}, b_{i}\right)$ and $\left(a_{i}^{\prime}, b_{i}^{\prime}\right)$ in $M$ and set $\alpha_{u}=0$ for all $u \in C_{i}$.

For every $e=(i, j) \in E_{H}$, where $i<j$, we do as follows:

- if $i \in U$ then include the edges $\left(s_{e}, t_{e}^{\prime}\right),\left(s_{e}^{\prime}, t_{e}\right)$, and $\left(s_{e}^{\prime \prime}, t_{e}^{\prime \prime}\right)$ in $M$; set $\alpha_{s_{e}}=\alpha_{t_{e}}=\alpha_{t_{e}^{\prime \prime}}=-1$ and $\alpha_{s_{e}^{\prime}}=\alpha_{t_{e}^{\prime}}=\alpha_{s_{e}^{\prime \prime}}=1$.
- else (so $j \in U$ ) include the edges $\left(s_{e}, t_{e}^{\prime \prime}\right),\left(s_{e}^{\prime}, t_{e}^{\prime}\right)$, and $\left(s_{e}^{\prime \prime}, t_{e}\right)$ in $M$; set $\alpha_{s_{e}}=\alpha_{t_{e}}=\alpha_{s_{e}^{\prime}}=-1$ and $\alpha_{t_{e}^{\prime}}=\alpha_{s_{e}^{\prime \prime}}=\alpha_{t_{e}^{\prime \prime}}=1$.
Set $\alpha_{a_{0}}=\alpha_{b_{0}}=0$. It can be checked that $\alpha_{u}+\alpha_{v}=0$ for every $(u, v) \in M$, hence $\sum_{u \in A \cup B} \alpha_{u}=$ 0 . Also $\alpha_{u}+\alpha_{v} \geq \operatorname{cost}_{M}(u, v)$ for every edge $(u, v)$ in $G$ and $\alpha_{u} \geq \operatorname{cost}_{M}(u, u)$ for all $u \in A \cup B$. Thus $\boldsymbol{\alpha}$ is a witness to $M$ 's popularity.

We will now calculate $w(M)$. The sum of edge weights in $M$ from vertices in $D_{e}$ is 5 , so this adds up to $5 m_{H}$ for all $e \in E_{H}$. For $i \in U$, the sum of edge weights in $M$ from vertices in $C_{i}$ is 2 , so this adds up to $2 k$ over all $i \in U_{H}$. For $j \notin U$, the sum of edge weights in $M$ from vertices in $C_{j}$ is 4 , so this adds adds up to $4\left(n_{H}-k\right)$ over all $j \notin U_{H}$. Thus $f(M)=5 m_{H}+4 n_{H}-2 k$.

We will now show the converse. Suppose $G$ has a popular matching of weight at least $5 m_{H}+$ $4 n_{H}-2 k$. Let $N$ be a max-weight popular matching in $G$. So $w(N) \geq 5 m_{H}+4 n_{H}-2 k$. Let $\boldsymbol{\alpha}$ be a witness of $N$ 's popularity and let $U_{N}=\left\{i \subseteq\left[n_{H}\right]: \alpha_{u}= \pm 1 \forall u \in C_{i}\right\}$. We know from Lemma 9 that $U_{N}$ is a vertex cover of $H$. We will now show that $\left|U_{N}\right| \leq k$.

We know from Claim 2 that for every $e \in E_{H}$, both $s_{e}$ and $t_{e}$ have to be matched in $N$. So each gadget $D_{e}$ contributes a weight of 5 towards $w(N)$. Each gadget $C_{i}$, for $i \in U_{N}$, contributes a weight of 2 towards $w(N)$ while each gadget $C_{j}$, for $j \notin U_{N}$, contributes a weight of 4 towards $w(N)$. Hence $w(N)=5 m_{H}+2\left|U_{N}\right|+4\left(n_{H}-\left|U_{N}\right|\right)$. Since $5 m_{H}+2\left|U_{N}\right|+4\left(n_{H}-\left|U_{N}\right|\right) \geq 5 m_{H}+4 n_{H}-2 k$, we get $\left|U_{N}\right| \leq k$. Thus $H$ has a vertex cover of size $k$.

Theorem 3 stated in Section 1 now follows. It is easy to see that Theorem 7 holds even if "popular matching of weight at least $5 m_{H}+4 n_{H}-2 k$ " is replaced by "max-size popular matching of weight at least $5 m_{H}+4 n_{H}-2 k$ ". Thus the problem of computing a max-size popular matching problem in $G=(A \cup B, E)$ of largest weight is also NP-hard.

## 6 Max-weight popular matchings: exact and approximate solutions

Let $G=(A \cup B, E)$ be an instance with a weight function $w: E \rightarrow \mathbb{R}_{\geq 0}$. We will now show that a popular matching in $G$ of weight at least OPT/2 can be computed in polynomial time, where OPT $=w\left(M^{*}\right)$ and $M^{*}$ is a max-weight popular matching in $G$. The following decomposition theorem for any popular matching $M$ in $G=(A \cup B, E)$ is known.

Theorem 8 ([7]). $M$ can be partitioned into $M_{0} \cup M_{1}$ such that $M_{0} \subseteq S$ and $M_{1} \subseteq D$, where $S$ is a stable matching and $D$ is a (strongly) dominant matching in $G$.

Any popular matching $M$ has a witness $\boldsymbol{\alpha} \in\{0, \pm 1\}^{n}$ : let $M_{0}$ (similarly, $M_{1}$ ) be the set of edges of $M$ on vertices $u$ with $\alpha_{u}=0$ (resp., $\alpha_{u}= \pm 1$ ). Since $\alpha_{a}+\alpha_{b}=0$ for all $(a, b) \in M$, the matching $M$ is a disjoint union of $M_{0}$ and $M_{1}$. If $M$ is a max-weight popular matching in $G$, then one of $M_{0}, M_{1}$ has weight at least $w(M) / 2$.

What the above result from [7] shows is that $M_{0}$ can be extended to a stable matching in $G$ and $M_{1}$ can be extended to a dominant matching in $G$. Consider the following algorithm.

1. Compute a max-weight stable matching $S^{*}$ in $G$.
2. Compute a max-weight dominant matching $D^{*}$ in $G$.
3. Return the matching in $\left\{S^{*}, D^{*}\right\}$ with larger weight.

Since all edge weights are non-negative, either the max-weight stable matching in $G$ or the maxweight dominant matching in $G$ has weight at least $w\left(M^{*}\right) / 2=\mathrm{OPT} / 2$. Thus Steps 1-3 compute a 2-approximation for max-weight popular matching in $G=(A \cup B, E)$.

Regarding the implementation of this algorithm, both $S^{*}$ and $D^{*}$ can be computed in polynomial time $25 / 7$. We also show descriptions of the stable matching polytope and the dominant matching polytope below. Thus the above algorithm runs in polynomial time.

### 6.1 A fast exponential time algorithm

We will now show an algorithm to compute a max-weight popular matching in $G=(A \cup B, E)$ with $w: E \rightarrow \mathbb{R}$. We will use an extended formulation from [21] of the popular fractional matching polytope $\mathcal{P}_{G}$. This is obtained by generalizing LP2 from Section 2 to all fractional matchings $\boldsymbol{x}$ in $G$, i.e., $\sum_{e \in \tilde{E}(u)} x_{e}=1$ for all vertices $u$ and $x_{e} \geq 0$ for all $e \in E$. So $\operatorname{cost}_{x}(a, b)$, which is the sum of votes of $a$ and $b$ for each other over their respective assignments in $\boldsymbol{x}$, replaces $\operatorname{cost}_{M}(a, b)$. Thus $\operatorname{cost}_{x}(a, b)=\sum_{b^{\prime}: b^{\prime} \prec_{a} b} x_{a b^{\prime}}-\sum_{b^{\prime}: b^{\prime} \succ{ }_{a} b} x_{a b^{\prime}}+\sum_{a^{\prime}: a^{\prime} \prec_{b} a} x_{a^{\prime} b}-\sum_{a^{\prime}: a^{\prime} \succ_{b} a} x_{a^{\prime} b}$.

Recall the subgraph $F_{G}=\left(A \cup B, E_{F}\right)$ where $E_{F}$ is the set of popular edges in $G$. The set $E_{F}$ can be efficiently computed as follows: call an edge stable if it belongs to some stable matching in $G$. It was shown in [7] that every popular edge in $G=(A \cup B, E)$ is a stable edge either in $G$ or in a larger bipartite graph $G^{\prime}$ on $O(n)$ vertices and $O(m+n)$ edges. All stable edges in $G$ and in $G^{\prime}$ can be identified in linear time [14].

Let $\mathcal{C}=\left\{\mathcal{C}_{1}, \ldots, \mathcal{C}_{h}\right\}$ be the set of connected components in $F_{G}$ and let $\ell$ be the number of nonsingleton sets in $\mathcal{C}$. Let us assume that components $\mathcal{C}_{1}, \ldots, \mathcal{C}_{\ell}$ have size at least 2 . For any popular matching $M$ with a witness $\boldsymbol{\alpha} \in\{0, \pm 1\}^{n}$, Lemma 6 tells us that $\boldsymbol{\alpha}$ defines a natural function (we will call this function $\alpha$ ) from $\mathcal{C}$ to $\{0,1\}$, where $\alpha\left(\mathcal{C}_{i}\right)=0$ if $\alpha_{u}=0$ for all $u \in \mathcal{C}_{i}$ and $\alpha\left(\mathcal{C}_{i}\right)=1$ if $\alpha_{u}= \pm 1$ for all $u \in \mathcal{C}_{i}$. Note that $\alpha\left(\mathcal{C}_{i}\right)=0$ for $\ell+1 \leq i \leq h$.

Definition 6. For $\boldsymbol{r}=\left(r_{1}, \ldots, r_{\ell}\right) \in\{0,1\}^{\ell}$, let $\mathcal{P}(\boldsymbol{r})$ be the convex hull of all popular matchings $M$ in $G$ such that $M$ has a witness $\boldsymbol{\alpha}$ with $\alpha\left(\mathcal{C}_{i}\right)=r_{i}$ for all $1 \leq i \leq \ell$.
$\mathcal{P}(\mathbf{0})$ is the stable matching polytope and $\mathcal{P}(\mathbf{1})$ is the dominant matching polytope. We will now show an extended formulation of $\mathcal{P}(\boldsymbol{r})$ for any $\boldsymbol{r} \in\{0,1\}^{\ell}$. First augment $\boldsymbol{r}$ with $r_{\ell+1}=\cdots=r_{h}=0$ so that $\boldsymbol{r} \in\{0,1\}^{h}$.
Our constraints. Let $U$ be the set of all unstable vertices in $\cup_{i} \mathcal{C}_{i}$ where $i \in\{1, \ldots, h\}$ is such that $r_{i}=0$. Let $A^{\prime}=A \backslash U$ and $B^{\prime}=B \backslash U$. It follows from Lemma 6 that all vertices in $U$ are left unmatched in any popular matching $M$ in $\mathcal{P}(\boldsymbol{r})$ while all vertices in $A^{\prime} \cup B^{\prime}$ are matched in $M$ (recall that all stable vertices of $G$ are matched in $M$ ). Consider constraints $\sqrt{2}$ - (6) given below.

$$
\begin{array}{rlrl}
\alpha_{a}+\alpha_{b} \geq \operatorname{cost}_{x}(a, b)+\left|r_{i}-r_{j}\right| & & \forall(a, b) \in E \cap\left(\mathcal{C}_{i} \times \mathcal{C}_{j}\right), 1 \leq i, j \leq h \\
\alpha_{a}+\alpha_{b} & =\operatorname{cost}_{x}(a, b) \quad & & \forall(a, b) \in E_{F} \\
\sum_{u \in A \cup B} \alpha_{u} & =0 \quad \text { and } \quad & -r_{i} \leq \alpha_{u} \leq r_{i} \quad \forall u \in \mathcal{C}_{i}, 1 \leq i \leq h \\
x_{(u, u)} & =1 \quad \forall u \in U \quad \text { and } \quad x_{(u, u)}=0 \quad \forall u \in A^{\prime} \cup B^{\prime} \\
\sum_{e \in \tilde{E}(u)} x_{e} & =1 \quad \forall u \in A \cup B, \quad x_{e} \geq 0 \quad \forall e \in E_{F}, \quad \text { and } \quad x_{e}=0 \quad \forall e \in E \backslash E_{F} \tag{6}
\end{array}
$$

The variables in the constraints above are $x_{e}$ for $e \in E$ and $\alpha_{u}$ for $u \in A \cup B$. Constraint (2) tightens the edge covering constraint $\alpha_{a}+\alpha_{b} \geq \operatorname{cost}_{x}(a, b)$ for edges in $\mathcal{C}_{i} \times \mathcal{C}_{j}$ with $r_{i} \neq r_{j}$. Consider any popular matching $M$ with witness $\boldsymbol{\alpha}$ such that $\alpha\left(\mathcal{C}_{i}\right) \neq \alpha\left(\mathcal{C}_{j}\right)$. So $M$ and $\boldsymbol{\alpha}$ satisfy $\alpha_{a}+\alpha_{b} \geq \operatorname{cost}_{M}(a, b)$. Since $\alpha_{a}+\alpha_{b} \in\{ \pm 1\}$ while $\operatorname{cost}_{M}(a, b) \in\{ \pm 2,0\}, M$ and $\boldsymbol{\alpha}$ have to satisfy $\alpha_{a}+\alpha_{b} \geq \operatorname{cost}_{M}(a, b)+1=\operatorname{cost}_{M}(a, b)+\left|r_{i}-r_{j}\right|$.

Constraint (3) makes the edge covering constraint tight for all popular edges $(a, b)$. This is because for any popular matching $M$ and witness $\boldsymbol{\alpha}$, we have $\alpha_{a}+\alpha_{b}=\operatorname{cost}_{M}(a, b)$ for any popular edge $(a, b)$ (see the proof of Lemma 5).

Constraint (6) is clearly satisfied by any popular matching $M$ and any witness $\boldsymbol{\alpha}$ satisfies $\sum_{u \in A \cup B} \alpha_{u}=0$. The other constraints in (4) and (5) are consequences of the parity $r_{i}$ of the component $\mathcal{C}_{i}$ that a vertex belongs to. We will prove the following theorem in Section 6.2.

Theorem 9. Constraints (2)-(6) define an extended formulation of the polytope $\mathcal{P}(\boldsymbol{r})$.
Thus a max-weight popular matching in $\mathcal{P}(\boldsymbol{r})$ can be computed in polynomial time and hence a max-weight popular matching in $G$ can be computed in $O^{*}\left(2^{\ell}\right)$ time by going through all $\boldsymbol{r} \in\{0,1\}^{\ell}$.

Recall that $\ell$ is the number of components in $F_{G}$ of size at least 2 . Since $\ell \leq n / 2$, this is an $O^{*}\left(2^{n / 2}\right)$ algorithm for computing a max-weight popular matching.

A faster exponential time algorithm. We will show in Section 6.3 that it is enough to go through all $\boldsymbol{r} \in\{0,1\}^{k}$, where $k$ is the number of components in $F_{G}$ of size at least 4 . We do this by introducing a new variable $p_{i}$, where $0 \leq p_{i} \leq 1$, to replace $r_{i}$ and represent the parity of $\mathcal{C}_{i}$, for each $\mathcal{C}_{i} \in \mathcal{C}$ of size 2 .

We will show that the resulting polytope is an extended formulation of the convex hull of all popular matchings $M$ in $G$ such that $M$ has a witness $\boldsymbol{\alpha}$ with $\alpha\left(\mathcal{C}_{i}\right)=r_{i}$ for $1 \leq i \leq k$. This yields an $O^{*}\left(2^{k}\right)$ algorithm for a max-weight popular matching in $G=(A \cup B, E)$. Since $k \leq n / 4$, this proves Theorem 4 stated in Section 1. Also, when $k=O(\log n)$, we have a polynomial time algorithm to compute a max-weight popular matching in $G$.

### 6.2 Proof of Theorem 9

Let $\mathcal{Q}^{\prime}(\boldsymbol{r}) \subseteq \mathbb{R}^{m+n}$ be the polytope defined by constraints $(2)-(6)$, where $|E|=m$. Let $\mathcal{Q}(\boldsymbol{r})$ denote the polytope $\mathcal{Q}^{\prime}(\boldsymbol{r})$ projected on to its first $m$ coordinates (those corresponding to $e \in E$ ). Our goal is to show that $\mathcal{Q}(\boldsymbol{r})$ is the polytope $\mathcal{P}(\boldsymbol{r})$.

We will first show that every popular matching $M$ with at least one witness $\boldsymbol{\alpha} \in\{0, \pm 1\}^{n}$ such that $\alpha\left(\mathcal{C}_{i}\right)=r_{i}$ for $i=1, \ldots, \ell$ belongs to $\mathcal{Q}(\boldsymbol{r})$. It is easy to see that $M$ and $\boldsymbol{\alpha}$ satisfy constraints (3)(6). Regarding constraint (2), for any edge $(a, b) \in \mathcal{C}_{i} \times \mathcal{C}_{j}$ such that $r_{i} \neq r_{j}$, we have $\alpha_{a}+\alpha_{b} \in\{ \pm 1\}$ while $\operatorname{cost}_{M}(a, b) \in\{ \pm 2,0\}$. Thus $\alpha_{a}+\alpha_{b} \geq \operatorname{cost}_{M}(a, b)$ implies that $\alpha_{a}+\alpha_{b} \geq \operatorname{cost}_{M}(a, b)+1$. Hence $(M, \boldsymbol{\alpha}) \in \mathcal{Q}^{\prime}(\boldsymbol{r})$, i.e., $M \in \mathcal{Q}(\boldsymbol{r})$. Thus $\mathcal{P}(\boldsymbol{r}) \subseteq \mathcal{Q}(\boldsymbol{r})$.

Let $(\boldsymbol{x}, \boldsymbol{\alpha}) \in \mathcal{Q}^{\prime}(\boldsymbol{r})$. We will now use the techniques and results from [27|18] to show that $\boldsymbol{x}$ is a convex combination of some popular matchings in $\mathcal{P}(\boldsymbol{r})$. This will prove $\mathcal{Q}(\boldsymbol{r})=\mathcal{P}(\boldsymbol{r})$.

The polytope $\mathcal{P}(\mathbf{0})$ is the stable matching polytope and a simple proof of integrality of Rothblum's formulation [25] of this polytope was given in [27]. When $G$ admits a perfect stable matching, the polytope $\mathcal{P}(\mathbf{1})$ is the same as $\mathcal{P}_{G}$ and a proof of integrality of $\mathcal{P}_{G}$ in this case was given in [18]. Note that $r_{1}, \ldots, r_{h}$ are all 0 in [27] and $r_{1}, \ldots, r_{h}$ are all 1 for this result in [18], i.e., the integrality of $\mathcal{P}_{G}$ when $G$ admits a perfect stable matching.

We will build a table $T$ of width 1 (as done in [27|18]) with $n^{\prime}=\left|A^{\prime} \cup B^{\prime}\right|$ rows: one corresponding to each vertex $u \in A^{\prime} \cup B^{\prime}$. The row corresponding to $u$ will be called $T_{u}$.

Form an array $X_{u}$ of length 1 as follows: if $x_{(u, v)}>0$ then there is a cell of length $x_{(u, v)}$ in $X_{u}$ with entry $v$ in it. The entries in $X_{u}$ will be sorted in increasing order of preference for $u \in A^{\prime}$ and in decreasing order of preference for $u \in B^{\prime}$. This was the order used in [27]. If $u \in \mathcal{C}_{i}$ where $i \in\{1, \ldots, \ell\}$ with $r_{i}=1$, then reorder $X_{u}$ as done in [18].

For any $a \in A^{\prime}$ that belongs to such a component $\mathcal{C}_{i}$, the initial or least preferred $\left(1+\alpha_{a}\right) / 2$ fraction of $X_{a}$ will be called the positive or blue sub-array of $X_{a}$ and the remaining part, which is the most preferred $\left(1-\alpha_{a}\right) / 2$ fraction of $X_{a}$, will be called the negative or red sub-array of $X_{a}$. The array $X_{a}$ will be reordered as shown in Fig. 3, i.e., the positive and negative sub-arrays of $X_{a}$ are swapped. Call the reordered array $T_{a}$.


Fig. 3. The array $X_{a}$ (on the left) will be reordered by swapping the positive and negative sub-arrays as shown above. The value $q_{a}=\left(1+\alpha_{a}\right) / 2$, so $1-q_{a}=\left(1-\alpha_{a}\right) / 2$.

A similar transformation from $X_{b}$ to $T_{b}$ was shown in [18] for each $b \in B^{\prime}$ that belongs to a component $\mathcal{C}_{i}$ where $i \in\{1, \ldots, \ell\}$ and $r_{i}=1$. The initial or most preferred $\left(1-\alpha_{b}\right) / 2$ fraction of $X_{b}$ will be called the negative sub-array of $X_{b}$ and the remaining part, which is the least preferred
$\left(1+\alpha_{b}\right) / 2$ fraction of $X_{b}$, will be called the positive sub-array of $X_{b}$. As before, swap the positive and negative sub-arrays of $X_{b}$ and call this reordered array $T_{b}$.

If $u \in A^{\prime} \cup B^{\prime}$ is in a component $\mathcal{C}_{i}$ with $r_{i}=0$, then we leave $X_{u}$ as it is. That is, we set $T_{u}=X_{u}$ (see Fig. 4). There are no positive or negative sub-arrays in $T_{u}$.


Fig. 4. The entries in $T_{u}$ are sorted in increasing order of preference for $u \in A$ and in decreasing order of preference for $u \in B$.

Finding the popular matchings that $\boldsymbol{x}$ is a convex combination of. Let $T$ be the table with rows $T_{u}$, for $u \in A^{\prime} \cup B^{\prime}$. For any $t \in[0,1)$, define the matching $M_{t}$ as follows:

- let $L(t)$ be the vertical line at distance $t$ from the left boundary of $T$;
- $L(t)$ intersects (or touches the left boundary of) some cell in $T_{u}$, call this cell $T_{u}[t]$, for each $u \in A^{\prime} \cup B^{\prime}$;

$$
\begin{equation*}
\text { define } M_{t}=\left\{(u, v): u \in A^{\prime} \cup B^{\prime} \text { and } v \text { is in cell } T_{u}[t]\right\} \tag{7}
\end{equation*}
$$

Validity of $M_{t}$. We need to prove that $M_{t}$ is a valid matching in $G$. That is, for any vertex $u \in A^{\prime} \cup B^{\prime}$, we need to show that if $v$ belongs to cell $T_{u}[t]$, then $u$ belongs to cell $T_{v}[t]$. Note that both $u$ and $v$ have to belong to the same component in the subgraph $F_{G}$ since $x_{u v}=0$ otherwise (since $x_{e}=0$ for $e \in E \backslash E_{F}$ ). Let $\mathcal{C}_{i}$ be the connected component in $F$ containing $u$ and $v$. There are 2 cases here: (i) $r_{i}=1$ and (ii) $r_{i}=0$.

The proof in case (i) follows directly from Theorem 3.2 in [18]. The proof in case (ii) is given in the proof of Theorem 1 in [27]. Both these proofs are based on the "tightness" of the edge $(u, v)$, i.e., $\alpha_{u}+\alpha_{v}=\operatorname{cost}_{x}(u, v)$ in case (i) and $\operatorname{cost}_{x}(u, v)=0$ in case (ii). The tightness of ( $u, v$ ) holds for us as well: since $x_{u v}>0,(u, v) \in E_{F}$, so $\alpha_{u}+\alpha_{v}=\operatorname{cost}_{x}(u, v)$ by constraint (3) in case (i); and in case (ii), we have $\alpha_{u}=\alpha_{v}=0$ by constraint (4) and so $\operatorname{cost}_{x}(u, v)=0$ by constraint (3).

Popularity of $M_{t}$. We will now show that $M_{t}$ is popular. Define a vector $\boldsymbol{\alpha}^{t} \in\{0, \pm 1\}^{n}$ :

- For $1 \leq i \leq h$ : if $r_{i}=0$ then set $\alpha_{u}^{t}=0$ for each $u \in \mathcal{C}_{i}$; else for each $u \in \mathcal{C}_{i}$
* if the cell $T_{u}[t]$ is positive (or blue) then set $\alpha_{u}^{t}=1$, else set $\alpha_{u}^{t}=-1$.

We will now show that $\boldsymbol{\alpha}^{t}$ is a witness of $M_{t}$. Thus $M_{t}$ will be a popular matching in $G$, in fact, $M_{t}$ will be in $\mathcal{P}(\boldsymbol{r})$. This is because by our assignment of $\alpha^{t}$-values above, $\alpha^{t}\left(\mathcal{C}_{i}\right)=r_{i}$ for $i \in\{1, \ldots, h\}$. Our first claim is that $\sum_{u \in A \cup B} \alpha_{u}^{t}=0$.

To prove the above claim, observe that for every vertex $u$ that is left unmatched in $M_{t}$ (all these vertices belong to $U$ ), we have $\alpha_{u}^{t}=0$. We now show that for every $(a, b)$ in $M_{t}$, we have $\alpha_{a}^{t}+\alpha_{b}^{t}=0$. Since $(a, b) \in E_{F}$, the vertices $a$ and $b$ belong to the same component $\mathcal{C}_{i}$. If $r_{i}=0$ then $\alpha_{a}^{t}=\alpha_{b}^{t}=0$ and hence $\alpha_{a}^{t}+\alpha_{b}^{t}=0$. If $r_{i}=1$ then $\alpha_{a}^{t}, \alpha_{b}^{t} \in\{ \pm 1\}$ and the proof that $\alpha_{a}^{t}+\alpha_{b}^{t}=0$ was shown in [18] (see Corollary 3.1). Thus $\sum_{u \in A \cup B} \alpha_{u}^{t}=0$.

We will now show that $\alpha_{u}^{t} \geq \operatorname{cost}_{M_{t}}(u, u)$ for all $u \in A \cup B$. Every vertex $u \in A^{\prime} \cup B^{\prime}$ is matched in $M_{t}$ and so $\operatorname{cost}_{M_{t}}(u, u)=-1$. Since $\alpha_{u}^{t} \geq-1$ for all $u \in A^{\prime} \cup B^{\prime}$, we have $\alpha_{u}^{t} \geq \operatorname{cost}_{M_{t}}(u, u)$ for these vertices. For $u \in U$, we have $\alpha_{u}^{t}=0=\operatorname{cost}_{M_{t}}(u, u)$.

What is left to show is that $\alpha_{a}^{t}+\alpha_{b}^{t} \geq \operatorname{cost}_{M_{t}}(a, b)$ for all $(a, b) \in E$. Lemma 10 below shows this. Hence we can conclude that $M_{t}$ is a popular matching in $G$, in particular, $M_{t} \in \mathcal{P}(\boldsymbol{r})$.

It is now easy to show that $\boldsymbol{x}$ is a convex combination of matchings that belong to $\mathcal{P}(\boldsymbol{r})$. To obtain these matchings, as done in [27], sweep a vertical line from the left boundary of table $T$ to its right boundary: whenever the line hits the left wall of one or more new cells, a new matching is obtained. If the left wall of the $i$-th leftmost cell(s) in the table $T$ is at distance $t_{i}$ from the left boundary of $T$, then we obtain the matching $M_{t_{i}}$ defined analogous to $M_{t}$ in (7).

Let $M_{0}, M_{t_{1}}, \ldots, M_{t_{d-1}}$ be all the matchings obtained by sweeping a vertical line through the table $T$. This means that $\boldsymbol{x}=t_{1} \cdot M_{0}+\left(t_{2}-t_{1}\right) \cdot M_{t_{1}}+\cdots+\left(1-t_{d-1}\right) \cdot M_{t_{d-1}}$. Thus $\boldsymbol{x}$ is a convex combination of matchings in $\mathcal{P}(\boldsymbol{r})$. This finishes the proof that $\mathcal{Q}(\boldsymbol{r})=\mathcal{P}(\boldsymbol{r})$.

Lemma 10. For any $(a, b) \in E$, we have $\alpha_{a}^{t}+\alpha_{b}^{t} \geq \operatorname{cost}_{M_{t}}(a, b)$.
Proof. Let $a \in \mathcal{C}_{i}$ and $b \in \mathcal{C}_{j}$, where $1 \leq i, j \leq h$. The proof that $\alpha_{a}^{t}+\alpha_{b}^{t} \geq \operatorname{cost}_{M_{t}}(a, b)$ when $r_{i}=r_{j}$ follows from 18/27. When $r_{i}=r_{j}=1$, it was shown in 18] (see Lemma 3.5) that $\alpha_{a}^{t}+\alpha_{b}^{t} \geq$ $\operatorname{cost}_{M_{t}}(a, b)$. When $r_{i}=r_{j}=0$, it was shown in [27] (see Theorem 1) that $\operatorname{cost}_{M_{t}}(a, b) \leq \alpha_{a}^{t}+\alpha_{b}^{t}=0$. We need to show $\alpha_{a}^{t}+\alpha_{b}^{t} \geq \operatorname{cost}_{M_{t}}(a, b)$ when $r_{i} \neq r_{j}$. Assume without loss of generality that $r_{i}=1$ and $r_{j}=0$. So $\alpha_{b}=0$.

The constraint corresponding to edge $(a, b)$ is $\alpha_{a}+\alpha_{b} \geq \operatorname{cost}_{x}(a, b)+\left|r_{i}-r_{j}\right|$ which simplifies to $\alpha_{a} \geq \operatorname{cost}_{x}(a, b)+1$. If $b \in U$ then $\operatorname{cost}_{x}(a, b) \geq 0$ since $b$ prefers $a$ to itself. So $\alpha_{a} \geq \operatorname{cost}_{x}(a, b)+1$ along with $\alpha_{a} \leq 1$ (by constraint (4)) implies that $\alpha_{a}=1$ and $\operatorname{cost}_{x}(a, b)=0$. This means that $\alpha_{a}^{t}=1$ and $a$ prefers all the entries in the array $T_{a}$ to $b$. Hence $\operatorname{cost}_{M_{t}}(a, b)=0$ and thus we have $\alpha_{a}^{t}+\alpha_{b}^{t}=1+0 \geq \operatorname{cost}_{M_{t}}(a, b)$.

Hence let us assume that $b \in B^{\prime}$. Since $\sum_{e \in \tilde{E}(u)} x_{e}=1$ for all $u, \operatorname{cost}_{x}(a, b)$ equals

$$
\sum_{b^{\prime} \prec_{a} b} x_{a b^{\prime}}-\sum_{b^{\prime} \succ_{a} b} x_{a b^{\prime}}+\sum_{a^{\prime} \prec_{b} a} x_{a^{\prime} b}-\sum_{a^{\prime} \succ_{b} a} x_{a^{\prime} b}=2\left(\sum_{b^{\prime} \prec_{a} b} x_{a b^{\prime}}+\sum_{a^{\prime} \prec_{b} a} x_{a^{\prime} b}+x_{a b}-1\right) .
$$

Here $x_{a b}=0$ since $a$ and $b$ belong to distinct components in $F_{G}$. So the constraint $\alpha_{a} \geq \operatorname{cost}_{x}(a, b)+1$ simplifies to:

$$
2 q_{a}-1 \geq 2\left(\sum_{b^{\prime} \prec_{a} b} x_{a b^{\prime}}+\sum_{a^{\prime} \prec_{b} a} x_{a^{\prime} b}-1\right)+1, \text { where } \alpha_{a}=2 q_{a}-1
$$

This becomes $q_{a} \geq \sum_{b^{\prime} \prec_{a} b} x_{a b^{\prime}}+\sum_{a^{\prime} \prec_{b} a} x_{a^{\prime} b}$. See Fig. 5 .


Fig. 5. The cells where either $a$ or $b$ gets matched to a neighbor worse than the other is restricted to the blue sub-array of $T_{a}$ and the sub-array of $T_{b}$ exactly below this (between the two dashed vertical lines). The sum of lengths of such cells added up over both $T_{a}$ and $T_{b}$ is at most $q_{a}$.

The neighbors with which $a$ gets paired in $\boldsymbol{x}$ start in increasing order of $a$ 's preference from the dashed line separating the red sub-array and blue sub-array in $T_{a}$ and this wraps around in left to right orientation (see Fig. 5). For $b$, the neighbors with which $b$ gets paired in $\boldsymbol{x}$ start in increasing order of $b$ 's preference from the right end of its array $T_{b}$ and this order is from right to left. Thus $b$ is matched in $\boldsymbol{x}$ to its worst neighbor at the right end of $T_{b}$ and to its best neighbor at the left end of $T_{b}$.

Since $\sum_{b^{\prime} \prec{ }_{a} b} x_{a b^{\prime}}+\sum_{a^{\prime} \prec_{b} a} x_{a^{\prime} b} \leq q_{a}$, this implies that the subarray where either $a$ is matched to a worse neighbor than $b$ (this subarray has length $\sum_{b^{\prime} \prec_{a} b} x_{a b^{\prime}}$ ) or $b$ is matched to a worse neighbor than $a$ (this subarray has length $\sum_{a^{\prime} \prec_{b} a} x_{a^{\prime} b}$ ) is confined to within the dashed line separating the red and blue subarrays of $T_{a}$ and the rightmost wall of $T_{b}$ (see Fig. 5). Also, there is no cell where both $a$ and $b$ are matched to worse neighbors than each other as this would make $\sum_{b^{\prime} \prec_{a} b} x_{a b^{\prime}}+\sum_{a^{\prime} \prec_{b} a} x_{a^{\prime} b}>q_{a}$.

If $T_{a}[t]$ is positive or blue, then $\alpha_{a}^{t}=1$. We have $\operatorname{cost}_{M_{t}}(a, b) \leq 0$ here since when one of $a, b$ is getting matched to a worse neighbor than the other, the other is getting matched to a better neighbor. Since $\alpha_{b}^{t}=0$ in the entire array, we have $\alpha_{a}^{t}+\alpha_{b}^{t}=1>\operatorname{cost}_{M_{t}}(a, b)$.

If $T_{a}[t]$ is negative or red, then $T_{a}[t]$ contains a neighbor that $a$ prefers to $b$ and similarly, $T_{b}[t]$ contains a neighbor that $b$ prefers to $a$. That is, $\operatorname{cost}_{M_{t}}(a, b)=-2$ and here we have $\alpha_{a}^{t}=-1$ and $\alpha_{b}^{t}=0$. Thus we have $\alpha_{a}^{t}+\alpha_{b}^{t}=-1>\operatorname{cost}_{M_{t}}(a, b)$. This shows that the edge $(a, b)$ is always covered by the sum of $\alpha^{t}$-values of $a$ and $b$.

### 6.3 A faster exponential time algorithm

Recall the popular subgraph $F_{G}=\left(A \cup B, E_{F}\right)$ and the set $\mathcal{C}=\left\{\mathcal{C}_{1}, \ldots, \mathcal{C}_{h}\right\}$ of connected components in $F_{G}$. Assume without loss of generality that $\mathcal{C}_{1}, \ldots, \mathcal{C}_{k}$ have size at least 4 and $\mathcal{C}_{k+1}, \ldots, \mathcal{C}_{\ell}$ have size 2 . Let us define the polytope $\mathcal{P}^{\prime}(\boldsymbol{r})$ as follows.

Definition 7. For $\boldsymbol{r}=\left(r_{1}, \ldots, r_{k}\right) \in\{0,1\}^{k}$, let $\mathcal{P}^{\prime}(\boldsymbol{r})$ be the convex hull of all popular matchings $M$ in $G$ such that $M$ has a witness $\boldsymbol{\alpha}$ with $\alpha\left(\mathcal{C}_{i}\right)=r_{i}$ for all $1 \leq i \leq k$.

Consider constraints (2)-(6) from Section 6. Regarding constraint (2), $r_{1}, \ldots, r_{k}$ are the coordinates in the vector $\boldsymbol{r} \in\{0,1\}^{k}$ while $r_{\ell+1}=\cdots=r_{h}=0$ as before, and $r_{i}=p_{i}$ for $k+1 \leq i \leq \ell$, where $p_{k+1}, \ldots, p_{\ell}$ are variables. So in constraint (2):
$-\left|r_{i}-r_{j}\right| \in\{0,1\}$ for $i, j \in\{1, \ldots, k\} \cup\{\ell+1, \ldots, h\}$.

- when one of $i, j$ (say, $i$ ) is in $\{k+1, \ldots, \ell\}$ and $j \in\{1, \ldots, k\} \cup\{\ell+1, \ldots, h\}$ then $\left|r_{i}-r_{j}\right|=1-p_{i}$ if $r_{j}=1$ and $\left|r_{i}-r_{j}\right|=p_{i}$ if $r_{j}=0$.
- when both $i, j$ are in $\{k+1, \ldots, \ell\}$, we replace constraint (2) with two constraints: one where $\left|r_{i}-r_{j}\right|$ is replaced by $p_{i}-p_{j}$ and another where $\left|r_{i}-r_{j}\right|$ is replaced by $p_{j}-p_{i}$.
Similarly, in constraint (4), $-r_{i} \leq \alpha_{u} \leq r_{i}$ now becomes $-p_{i} \leq \alpha_{u} \leq p_{i}$, for $u \in \mathcal{C}_{i}$ where $i \in\{k+1, \ldots, \ell\}$. Constraints (3), (5), and (6) remain the same as before. Also, the sets $U, A^{\prime}, B^{\prime}$ are the same as before since all in $\mathcal{C}_{k+1} \cup \cdots \cup \mathcal{C}_{\ell}$ are stable vertices. We will show the following theorem here. The proof of Theorem 10 will follow the same outline as the proof of Theorem 9.

Theorem 10. The revised constraints (2)-(6) along with the constraints $0 \leq p_{i} \leq 1$ for $k+1 \leq i \leq \ell$ define an extended formulation of the polytope $\mathcal{P}^{\prime}(\boldsymbol{r})$.

Proof. Let $\mathcal{S}^{\prime}(\boldsymbol{r})$ be the polytope defined by the revised constraints (2)-(6) along with $0 \leq p_{i} \leq 1$ for $k+1 \leq i \leq \ell$. Let $\mathcal{S}(\boldsymbol{r})$ denote the polytope $\mathcal{S}^{\prime}(\boldsymbol{r})$ projected on to the coordinates corresponding to $e \in E$. We will now show that $\mathcal{S}(\boldsymbol{r})$ is the polytope $\mathcal{P}^{\prime}(\boldsymbol{r})$.

It is easy to see that every popular matching $M$ with at least one witness $\boldsymbol{\alpha}$ such that $\alpha\left(\mathcal{C}_{i}\right)=r_{i}$ for $i=1, \ldots, \ell$ belongs to $\mathcal{S}(\boldsymbol{r})$. Thus $\mathcal{P}^{\prime}(\boldsymbol{r}) \subseteq \mathcal{S}(\boldsymbol{r})$. Let $(\boldsymbol{x}, \boldsymbol{\alpha}, \boldsymbol{p}) \in \mathcal{S}^{\prime}(\boldsymbol{r})$. We will now show that $\boldsymbol{x}$ is a convex combination of matchings in $\mathcal{P}^{\prime}(\boldsymbol{r})$.

As done in the proof of Theorem9, we will construct a table $T$ with $n^{\prime}=\left|A^{\prime} \cup B^{\prime}\right|$ rows. The rows of vertices outside $\mathcal{C}_{k+1} \cup \cdots \cup \mathcal{C}_{\ell}$ will be the same as before. Recall that $\left|\mathcal{C}_{i}\right|=2$ for $k+1 \leq i \leq \ell$ and for any $i \in\{k+1, \ldots, \ell\}$ : every vertex $u \in \mathcal{C}_{i}$ is matched to its only neighbor $v \in \mathcal{C}_{i}$ with $x_{u v}=1$. So we do not reorder $X_{u}$ as it consists of just a single entry, i.e., $T_{u}=X_{u}$. However we partition $T_{u}$ into 3 cells: (i) the positive or blue cell, (ii) the negative or red cell, and (iii) the zero or green cell.

- for $a \in \mathcal{C}_{i}$, compute $2 s_{i}=1-p_{i}$. Since $p_{i} \leq 1$, we have $s_{i} \geq 0$. Also $p_{i} \geq \pm \alpha_{u}$ since $\alpha_{u}$ is sandwiched between $-p_{i}$ and $p_{i}$. So $1-2 s_{i} \geq \pm \alpha_{a}$, i.e., $s_{i} \leq q_{a}$ and $s_{i} \leq 1-q_{a}$ where $q_{a}=\left(1+\alpha_{a}\right) / 2$. The array $T_{a}$ gets divided into three cells as shown in Fig. 6.
The leftmost $1-q_{a}-s_{i}$ part of $T_{a}$ is its "negative" (or red) cell and the rightmost $q_{a}-s_{i}$ part is its "positive" (or blue) cell. The part left in between the positive and negative parts, which is of length $2 s_{i}$, is its "zero" (or green) cell. All 3 cells contain the same vertex $b^{*}$ where $\mathcal{C}_{i}=\left\{a, b^{*}\right\}$.


Fig. 6. The red part (of length $1-q_{a}-s_{i}$ ) is the negative cell of $T_{a}$ and the blue part (of length $q_{a}-s_{i}$ ) is the positive cell of $T_{a}$. The middle or green part (of length $2 s_{i}$ ) is its zero cell.


Fig. 7. The blue part (of length $1-q_{b}-s_{i}$ ) is the positive cell of $T_{b}$ and the red part (of length $q_{b}-s_{i}$ ) is the negative cell of $T_{b}$. The middle or green part (of length $2 s_{i}$ ) is its zero cell.

- for $b \in \mathcal{C}_{i}$, compute $2 s_{i}=1-p_{i}$. Since $\pm \alpha_{b} \leq p_{i} \leq 1$, we have $s_{i} \geq 0$; also $s_{i} \leq q_{b}$ and $s_{i} \leq 1-q_{b}$ where $q_{b}=\left(1-\alpha_{b}\right) / 2$. The array $T_{b}$ gets divided into three cells as shown in Fig. 7 .
The leftmost $1-q_{b}-s_{i}$ part of $T_{b}$ is its "positive" (or blue) cell and the rightmost $q_{b}-s_{i}$ part is its "negative" (or red) cell. The part left in between these two parts, which is of length $2 s_{i}$, is its "zero" (or green) cell. All 3 cells contain the same vertex $a^{*}$ where $\mathcal{C}_{i}=\left\{a^{*}, b\right\}$.

In order to find the popular matchings that $\boldsymbol{x}$ is a convex combination of, we build the table $T$ and define $M_{t}$ exactly as done in the proof of Theorem 9 . It follows from the same arguments as before that $M_{t}$ is a matching. In particular, for $k+1 \leq i \leq \ell$, we have $\left|\mathcal{C}_{i}\right|=2$, say $\mathcal{C}_{i}=\{u, v\}$, and so $v$ is the only entry in the entire array $T_{u}$ and similarly, $u$ is the only entry in the entire array $T_{v}$.

We now need to show that $M_{t}$ is popular. For this, we define a vector $\boldsymbol{\alpha}^{t} \in\{0, \pm 1\}^{n}$ : for $i \in\{1, \ldots, k\} \cup\{\ell+1, \ldots, h\}$, the assignment of $\alpha_{u}^{t}$-values is exactly the same as in the proof of Theorem 9. For $i \in\{k+1, \ldots, \ell\}$ and each $u \in \mathcal{C}_{i}$ do:

- set $\alpha_{u}^{t}=1$ if the cell $T_{u}[t]$ is positive (or blue)
- set $\alpha_{u}^{t}=0$ if the cell $T_{u}[t]$ is zero (or green)
- set $\alpha_{u}^{t}=-1$ if the cell $T_{u}[t]$ is negative (or red)

We will now show that $\sum_{u \in A \cup B} \alpha_{u}^{t}=0$. The only new step is to show that $\alpha_{a}^{t}+\alpha_{b}^{t}=0$ for $(a, b) \in M_{t}$, where $\mathcal{C}_{i}=\{a, b\}$, i.e., $i \in\{k+1, \ldots, \ell\}$. Here we have $\alpha_{a}+\alpha_{b}=\operatorname{cost}_{x}(a, b)=0$ and this is because $x_{a b}=1$.

So $q_{a}=\left(1+\alpha_{a}\right) / 2=\left(1-\alpha_{b}\right) / 2=q_{b}$ and this implies the length of the positive (or blue) cell in $T_{a}$, which is $q_{a}-s_{i}$ (see Fig. 6), equals the length of the negative (or red) cell in $T_{b}$, which is $q_{b}-s_{i}$, (see Fig. 7).

- Hence the positive cell of $T_{a}$ is perfectly aligned with the negative cell of $T_{b}$. So if $L(t)$ goes through the positive cell of $T_{a}$, i.e. if $\alpha_{a}^{t}=1$, then $\alpha_{b}^{t}=-1$.
- Similarly, the negative cell of $T_{a}$, which is of length $1-q_{a}-s_{i}$, is perfectly aligned with the positive cell of $T_{b}$, which is of length $1-q_{b}-s_{i}$. So if $L(t)$ goes through the negative cell of $T_{a}$, i.e. if $\alpha_{a}^{t}=-1$, then $\alpha_{b}^{t}=1$.
- Thus the zero cells in $T_{a}$ and $T_{b}$ are perfectly aligned with each other. So if $L(t)$ goes through the zero cell of $T_{a}$, i.e. if $\alpha_{a}^{t}=0$, then $\alpha_{b}^{t}=0$.

Hence $\alpha_{a}^{t}+\alpha_{b}^{t}=0$ for all $(a, b) \in M_{t}$ and so $\sum_{u \in A \cup B} \alpha_{u}^{t}=0$.
It is easy to see that $\alpha_{u}^{t} \geq \operatorname{cost}_{M_{t}}(u, u)$ for all $u \in A \cup B$. What is left to show is that $\alpha_{a}^{t}+\alpha_{b}^{t} \geq$ $\operatorname{cost}_{M_{t}}(a, b)$ for all $(a, b) \in E$. Let $a \in \mathcal{C}_{i}$ and $b \in \mathcal{C}_{j}$. When both $i$ and $j$ are in $\{1, \ldots, k\} \cup\{\ell+$
$1, \ldots, h\}$, the proof of Lemma 10 shows that the edge covering constraint holds. Lemmas 11 and 12 below show that the edge covering constraint holds when one or both the indices are in $\{k+1, \ldots, \ell\}$.

This completes the proof that $M_{t}$ is a popular matching in $G$, in particular, $M_{t} \in \mathcal{P}^{\prime}(\boldsymbol{r})$. The rest of the argument that $\boldsymbol{x}$ is a convex combination of matchings that belong to $\mathcal{P}^{\prime}(\boldsymbol{r})$ is exactly the same as given in the proof of Theorem 9 . Thus we can conclude that $\mathcal{S}(\boldsymbol{r})=\mathcal{P}^{\prime}(\boldsymbol{r})$.

Lemma 11. Let $(a, b) \in E$ with $a \in \mathcal{C}_{i}$ and $b \in \mathcal{C}_{j}$. Suppose one of $i, j$ is in $\{1, \ldots, k\} \cup\{\ell+1, \ldots, h\}$ and the other is in $\{k+1, \ldots, \ell\}$. Then $\alpha_{a}^{t}+\alpha_{b}^{t} \geq \operatorname{cost}_{M_{t}}(a, b)$.

Proof. Assume without loss of generality that $i \in\{k+1, \ldots, \ell\}$. So $\mathcal{C}_{i}=\left\{a, b^{*}\right\}$. Since $a$ and $b$ are in different components in $F_{G}$, the edge $(a, b) \notin E_{F}$ and so $x_{a b}=0$. The index $j \in\{1, \ldots, k\} \cup\{\ell+$ $1, \ldots, h\}$, so $r_{j}$ is 0 or 1 .
Case 1. Suppose $r_{j}=0$. So $\alpha_{b}=0$. Suppose $b \in U$, i.e., $b$ is unmatched in $\boldsymbol{x}$. Then the constraint $\alpha_{a}+\alpha_{b} \geq \operatorname{cost}_{x}(a, b)+\left|r_{i}-r_{j}\right|$ for $(a, b)$ becomes $\alpha_{a} \geq p_{i}$ since $\operatorname{cost}_{x}(a, b)=0$. We also have $\alpha_{a} \leq p_{i}$ (the revised constraint (4)). Hence $\alpha_{a}=p_{i}$ and this means $2 s_{i}=1-\alpha_{a}$, i.e., $s_{i}=\left(1-\alpha_{a}\right) / 2=1-q_{a}$ (see Fig. 66. Then there is no negative (or red) cell in the entire array $T_{a}$. In other words, $\alpha_{a}^{t} \geq 0$ throughout the array $T_{a}$ and so $\alpha_{a}^{t}+\alpha_{b}^{t} \geq 0=\operatorname{cost}_{M_{t}}(a, b)$.

Hence let us assume that $b \in B^{\prime}$. So $\operatorname{cost}_{x}(a, b)=2\left(\sum_{b^{\prime} \prec_{a} b} x_{a b^{\prime}}+\sum_{a^{\prime} \prec_{b} a} x_{a^{\prime} b}-1\right)$. Thus the constraint $\alpha_{a}+\alpha_{b}-\left|r_{i}-r_{j}\right| \geq \operatorname{cost}_{x}(a, b)$ becomes: $\left(\right.$ where $\left.\alpha_{a}=2 q_{a}-1\right)$

$$
\begin{aligned}
\left(2 q_{a}-1\right)-p_{i} & \geq 2\left(\sum_{b^{\prime} \prec_{a} b} x_{a b^{\prime}}+\sum_{a^{\prime} \prec_{b} a} x_{a^{\prime} b}-1\right) \\
q_{a}+\left(1-p_{i}\right) / 2 & \geq \sum_{b^{\prime} \prec_{a} b} x_{a b^{\prime}}+\sum_{a^{\prime} \prec_{b} a} x_{a^{\prime} b} \\
q_{a}+s_{i} & \geq \sum_{b^{\prime} \prec_{a} b} x_{a b^{\prime}}+\sum_{a^{\prime} \prec_{b} a} x_{a^{\prime} b} \quad\left(\text { since } s_{i}=\left(1-p_{i}\right) / 2\right)
\end{aligned}
$$

Note that $q_{a}+s_{i}$ is the sum of lengths of the positive (or blue) and zero (or green) cells of $T_{a}$ (see Fig. 8) and so $q_{a}+s_{i} \leq 1$. In these two cells of $T_{a}, \alpha_{a}^{t}$ is either 1 or 0 .


Fig. 8. The vertex $a$ is matched to the same partner in the entire array $T_{a}$ and $b$ 's increasing order of partners in $\boldsymbol{x}$ starts from the right end of its array $T_{b}$.

Suppose $a$ prefers $b^{*}$ to $b$. Then $\sum_{b^{\prime} \prec_{a} b} x_{a b^{\prime}}=0$, so $q_{a}+s_{i} \geq \sum_{a^{\prime} \prec_{b} a} x_{a^{\prime} b}$. So while $b$ is matched to a neighbor worse than $a, \alpha_{a}^{t} \geq 0$. Note that $\alpha_{b}^{t}=0$ throughout the array $T_{b}$. Thus $\alpha_{a}^{t}+\alpha_{b}^{t} \geq 0 \geq$ $\operatorname{cost}_{M_{t}}(a, b)$.

Suppose $a$ prefers $b$ to $b^{*}$. Then $\sum_{b^{\prime} \prec_{a} b} x_{a b^{\prime}}=1$ and so $q_{a}+s_{i} \geq 1$. This means $q_{a}+s_{i}=1$ and so $\sum_{a^{\prime} \prec_{b} a} x_{a^{\prime} b}=0$. That is, $b$ prefers each of its partners in the array $T_{b}$ to $a$ and so $\operatorname{cost}_{M_{t}}(a, b)=0$. Also $q_{a}+s_{i}=1$ implies that there is no negative (or red) cell in $T_{a}$. Thus $\alpha_{a}^{t} \geq 0$ throughout the array $T_{a}$ and so $\alpha_{a}^{t}+\alpha_{b}^{t} \geq 0=\operatorname{cost}_{M_{t}}(a, b)$.

Case 2. Suppose $r_{j}=1$. So $\alpha_{b}=1-2 q_{b}$ and the constraint $\alpha_{a}+\alpha_{b}-\left|r_{i}-r_{j}\right| \geq \operatorname{cost}_{x}(a, b)$ becomes:

$$
\begin{aligned}
\left(2 q_{a}-1\right)+\left(1-2 q_{b}\right)-\left(1-p_{i}\right) & \geq 2\left(\sum_{b^{\prime} \prec_{a} b} x_{a b^{\prime}}+\sum_{a^{\prime} \prec_{b} a} x_{a^{\prime} b}-1\right) \\
q_{a}-q_{b}-\left(1-p_{i}\right) / 2 & \geq \sum_{b^{\prime} \prec_{a} b} x_{a b^{\prime}}+\sum_{a^{\prime} \prec_{b} a} x_{a^{\prime} b}-1 \\
\left(q_{a}-s_{i}\right)+\left(1-q_{b}\right) & \geq \sum_{b^{\prime} \prec_{a} b} x_{a b^{\prime}}+\sum_{a^{\prime} \prec_{b} a} x_{a^{\prime} b} .
\end{aligned}
$$

Note that $q_{a}-s_{i}$ is the length of the positive (or blue) cell in $T_{a}$ (see Fig. 9). Similarly, $1-q_{b}$ is the length of the blue sub-array of $T_{b}$.

Suppose $a$ prefers $b^{*}$ to $b$. Then $\sum_{b^{\prime} \prec_{a} b} x_{a b^{\prime}}=0$ and so $\sum_{a^{\prime} \prec_{b} a} x_{a^{\prime} b} \leq\left(q_{a}-s_{i}\right)+\left(1-q_{b}\right)$ and this is the sum of lengths of blue sub-arrays of $T_{a}$ and $T_{b}$. Consider traversing the array $T_{b}$ starting from the dashed line separating its blue sub-array from its red sub-array in a right-to-left orientation that wraps around. The sum of length of the cells where $b$ is matched to neighbors worse than $a$ is $\sum_{a^{\prime} \prec_{b} a} x_{a^{\prime} b}$. This is at most the sum of lengths of blue sub-arrays in $T_{a}$ and $T_{b}$. Thus while $b$ is matched to a neighbor worse than $a$, we have $\alpha_{a}^{t}+\alpha_{b}^{t} \geq 0=\operatorname{cost}_{M_{t}}(a, b)$ and when $b$ is matched to a neighbor better than $a$, we have $\alpha_{a}^{t}+\alpha_{b}^{t} \geq-2=\operatorname{cost}_{M_{t}}(a, b)$ since $\alpha_{a}^{t} \geq-1$ and $\alpha_{b}^{t} \geq-1$.


Fig. 9. The vertex $a$ is matched to the same partner in the entire array $T_{a}$ and $b$ 's increasing order of partners in $T_{b}$ starts from the dashed line in right to left orientation.

Suppose $a$ prefers $b$ to $b^{*}$. Then $\sum_{b^{\prime} \prec_{a} b} x_{a b^{\prime}}=1$ and so $\sum_{a^{\prime} \prec_{b} a} x_{a^{\prime} b} \leq\left(q_{a}-s_{i}\right)+\left(1-q_{b}\right)-1$. Since $\sum_{a^{\prime} \prec_{b} a} x_{a^{\prime} b} \geq 0$, this means the sum of lengths of blue sub-arrays in $T_{a}$ and $T_{b}$ is at least 1 and $\sum_{a^{\prime} \prec_{b} a} x_{a^{\prime} b}$ is bounded by how much $\left(q_{a}-s_{i}\right)+\left(1-q_{b}\right)$ exceeds 1 . Since the blue sub-array in $T_{b}$ begins from its left end and the blue cell in $T_{a}$ starts from its right end, for any $t \in[0,1)$, at least one of the cells $T_{a}[t], T_{b}[t]$ is blue. Moreover, when $b$ prefers $a$ to its neighbors in $T_{b}$, then both $T_{a}[t]$ and $T_{b}[t]$ are blue. So while $b$ is matched to a neighbor better than $a$, we have $\alpha_{a}^{t}+\alpha_{b}^{t} \geq 0=\operatorname{cost}_{M_{t}}(a, b)$ and when $b$ is matched to a neighbor worse than $a$, we have $\alpha_{a}^{t}+\alpha_{b}^{t}=2=\operatorname{cost}_{M_{t}}(a, b)$.

Lemma 12. Let $(a, b)$ be an edge in $\mathcal{C}_{i} \times \mathcal{C}_{j}$, where $i, j \in\{k+1, \ldots, \ell\}$. Then $\alpha_{a}^{t}+\alpha_{b}^{t} \geq \operatorname{cost}_{M_{t}}(a, b)$.

Proof. Here both $\mathcal{C}_{i}$ and $\mathcal{C}_{j}$ have size 2. When $i=j$, we showed that $\alpha_{a}^{t}+\alpha_{b}^{t}=0=\operatorname{cost}_{M_{t}}(a, b)$ in the proof of $\sum_{u \in A \cup B} \alpha_{u}^{t}=0$.

So we now assume $i \neq j$ and so $x_{a b}=0$. The constraint $\alpha_{a}+\alpha_{b}-\left|r_{i}-r_{j}\right| \geq \operatorname{cost}_{x}(a, b)$ becomes the following two constraints, where $\alpha_{a}=2 q_{a}-1$ and $\alpha_{b}=1-2 q_{b}$.

$$
\begin{aligned}
& \left(2 q_{a}-1\right)+\left(1-2 q_{b}\right)-\left(p_{i}-p_{j}\right) \geq 2\left(\sum_{b^{\prime} \prec_{a} b} x_{a b^{\prime}}+\sum_{a^{\prime} \prec_{b} a} x_{a^{\prime} b}-1\right) \text { and } \\
& \left(2 q_{a}-1\right)+\left(1-2 q_{b}\right)+\left(p_{i}-p_{j}\right) \geq 2\left(\sum_{b^{\prime} \prec_{a} b} x_{a b^{\prime}}+\sum_{a^{\prime} \prec_{b} a} x_{a^{\prime} b}-1\right) .
\end{aligned}
$$

Let $s_{i}=\left(1-p_{i}\right) / 2$ and $s_{j}=\left(1-p_{j}\right) / 2$. The above two constraints imply the following two constraints respectively:

$$
\begin{align*}
& \left(q_{a}+s_{i}\right)+\left(1-q_{b}-s_{j}\right) \geq \sum_{b^{\prime} \prec_{a} b} x_{a b^{\prime}}+\sum_{a^{\prime} \prec_{b} a} x_{a^{\prime} b}  \tag{8}\\
& \left(q_{a}-s_{i}\right)+\left(1-q_{b}+s_{j}\right) \geq \sum_{b^{\prime} \prec_{a} b} x_{a b^{\prime}}+\sum_{a^{\prime} \prec_{b} a} x_{a^{\prime} b} \tag{9}
\end{align*}
$$

Note that the length of the blue cell in $T_{a}$ is $q_{a}-s_{i}$ while $q_{a}+s_{i}$ is the sum of the lengths of the blue and green cells in $T_{a}$ (see the top 2 arrays in Fig. 10. Similarly, the length of the blue cell in $T_{b}$ is $1-q_{b}-s_{j}$ while $1-q_{b}+s_{j}$ is the sum of the lengths of the blue and green cells in $T_{b}$ (see the bottom 2 arrays in Fig. 10. We will consider 3 cases here.
Case 1. Both $a$ and $b$ prefer their partners in $\boldsymbol{x}$ to each other. This is the easiest case. Here $\operatorname{cost}_{M_{t}}(a, b)=-2$ and since $\alpha_{u}^{t} \geq-1$ for all vertices $u$, we have $\alpha_{a}^{t}+\alpha_{b}^{t} \geq-2=\operatorname{cost}_{M_{t}}(a, b)$.


Fig. 10. The length of the blue cell in $T_{a}$ (similarly, $T_{b}$ ) is $q_{a}-s_{i}$ (resp., $1-q_{b}-s_{j}$ ). The length of the blue + green cells in $T_{a}$ (similarly, $T_{b}$ ) is $q_{a}+s_{i}$ (resp., $1-q_{b}+s_{j}$ ).

Case 2. Exactly one of $a, b$ prefers its partner in $\boldsymbol{x}$ to the other. So $\operatorname{cost}_{M_{t}}(a, b)=0$ and the right side of constraints (8) and (9) is 1.

Constraint (8) means that the length of the (blue + green) cells of $T_{a}$ added to the length of the blue cell of $T_{b}$ is at least 1 . Hence the length of the blue cell of $T_{b}$ is at least the length of the red cell in $T_{a}$ (see Fig. 10 ).

Constraint (9) means that the length of the (blue + green) cells of $T_{b}$ added to the length of the blue cell of $T_{a}$ is at least 1. Hence the length of the blue cell of $T_{a}$ is at least the length of the red cell in $T_{b}$ (see Fig. 10 ).

Thus for any $t \in\left[0,1\right.$ ), it is the case that either (i) at least one of $T_{a}[t], T_{b}[t]$ is blue or (ii) both the cells $T_{a}[t]$ and $T_{b}[t]$ are green. Thus we have $\alpha_{a}^{t}+\alpha_{b}^{t} \geq 0=\operatorname{cost}_{M_{t}}(a, b)$.
Case 3. Both $a$ and $b$ prefer each other to their partners in $\boldsymbol{x}$. So $\operatorname{cost}_{M_{t}}(a, b)=2$ and the right side of constraints (8) and (9) is also 2.

Since each of $\left(q_{a}+s_{i}\right),\left(q_{a}-s_{i}\right),\left(1-q_{b}+s_{j}\right),\left(1-q_{b}-s_{j}\right)$ is at most 1 , it follows that $q_{a}-s_{i}=1$ and $1-q_{b}-s_{j}=1$. Thus the entire array $T_{a}$ is blue and similarly, the entire array $T_{b}$ is also blue. Hence $\alpha_{a}^{t}=\alpha_{b}^{t}=1$. Thus we have $\alpha_{a}^{t}+\alpha_{b}^{t}=2=\operatorname{cost}_{M_{t}}(a, b)$.

Acknowledgment. Thanks to Chien-Chung Huang for useful discussions on strongly dominant matchings.

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## Appendix

Mixed matchings. Consider the roommates instance $G$ that is a triangle on the 3 vertices $a, b, c$ with the following preference lists: $a$ prefers $b$ to $c$ while $b$ prefers $c$ to $a$ and $c$ prefers $a$ to $b$. This instance has no mixed matching that is stable. A mixed matching $\Pi$ is equivalent to a point $\boldsymbol{x}$ in
the matching polytope of $G$. A mixed matching $\Pi$ is stable if the corresponding point $\boldsymbol{x}$ satisfies the following stability constraint for every edge $(u, v)$ :

$$
x_{u v}+\sum_{v^{\prime}: v^{\prime} \succ_{u} v} x_{u v^{\prime}}+\sum_{u^{\prime}: u^{\prime} \succ_{v} u} x_{u^{\prime} v} \geq 1 .
$$

It can be checked that there is no point $\boldsymbol{x}$ in the matching polytope of $G$ that satisfies the stability constraints for all edges. The mixed matching $\Pi=\left\{\left(M_{1}, 1 / 3\right),\left(M_{2}, 1 / 3\right),\left(M_{3}, 1 / 3\right)\right\}$, where $M_{1}=$ $\{(a, b)\}, M_{2}=\{(b, c)\}, M_{3}=\{(c, a)\}$, is popular in this instance.

## Irving's algorithm

Given a roommates instance $G=(V, E)$ with strict preferences, Irving's algorithm [19] determines if $G$ admits a stable matching or not and if so, returns one. Irving's algorithm assumed $G$ to be a complete graph, however the algorithm easily generalizes to non-complete graphs as well and hence we will not assume $G$ to be complete.

Irving's algorithm consists of 2 phases:

1. In the first phase, we consider the bipartite graph $G^{*}=\left(V \cup V^{\prime}, E^{\prime}\right)$ where $V^{\prime}=\left\{u^{\prime}: u \in V\right\}$. So $G^{*}$ has 2 copies $u$ and $u^{\prime}$ of each vertex $u \in V$, one on either side of the graph. The edge set $E^{\prime}=\left\{\left(u, v^{\prime}\right):(u, v) \in E\right\}$.
Run Gale-Shapley algorithm on $G^{*}$ with vertices in $V$ proposing and those in $V^{\prime}$ disposing. Let $M^{*}$ be the resulting matching. If $\left(u, v^{\prime}\right) \in M^{*}$ then prune the edge set $E$ of $G$ as follows:

- delete all neighbors ranked worse than $u$ from the preference list of $v$
- make the adjacency lists symmetric so that if $v$ deletes $u$ from its list then $u$ also deletes $v$ from its list.

2. If the reduced adjacency list of every vertex which received at least 1 proposal consists of a single neighbor then the resulting edge set $E$ defines a stable matching $M$. Else the adjacency lists are further reduced by eliminating "rotations".

- A rotation $R=\left\{\left(a_{0}, b_{0}\right),\left(a_{1}, b_{1}\right), \ldots,\left(a_{k-1}, b_{k-1}\right)\right\}$ is a set of edges such that for all $i \in$ $\{0, \ldots, k-1\}$, the vertex $b_{i}$ is $a_{i}$ 's most preferred neighbor in its reduced preference list (thus $a_{i}$ would be $b_{i}$ 's least preferred neighbor in its reduced preference list); moreover, the second person on $a_{i}$ 's reduced preference list is $b_{i+1}$ (here $b_{k}=b_{0}$ ).
- The second phase of Irving's algorithm identifies such rotations and deletes them. The crucial observation here is that if $G$ admits a stable matching then so does $G \backslash R$.
- This step of eliminating rotations continues till either the updated reduced adjacency list of every vertex consists of a single neighbor or the updated reduced adjacency list of some vertex that received at least 1 proposal in the first phase is empty. In the former case, the resulting edge set is a stable matching and in the latter case, $G$ has no stable matching.

Consider Irving's algorithm in the roommates instance $G^{\prime}$ that corresponds to the instance $G$ on 4 vertices $a, b, c, d$ described in Section 1. The reduced adjacency lists at the end of the first phase are as follows:

$$
a: c^{-} \succ d^{-} \quad b: d^{-} \succ c^{+} \quad c: b^{-} \succ a^{+} \quad d: a^{+} \succ b^{+}
$$

Eliminating the rotation $\left\{\left(a^{+}, c^{-}\right),\left(b^{+}, d^{-}\right)\right\}$yields the matching $M_{1}^{\prime}=\left\{\left(a^{+}, d^{-}\right),\left(b^{-}, c^{+}\right)\right\}$. Instead, we could have eliminated the rotation $\left\{\left(c^{+}, b^{-}\right),\left(d^{-}, a^{+}\right)\right\}$. This yields the matching $M_{2}^{\prime}=$ $\left\{\left(a^{+}, c^{-}\right),\left(b^{+}, d^{-}\right)\right\}$.

